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Analysis of Bifurcation and Chaos of the Piezoelectric Plate including Damage Effects

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Abstract

This paper presents a constitutive model for piezoelectric materials containing a substantive of distributed cracks. The model is formulated in a continuum damage mechanics framework using internal variables taken as second rank tensors. Based on the Talreja's tensor valued internal state damage variables as well as the Helmhotlz free energy of piezoelectric materials, the constitutive model is applied to analysis of bifurcation and chaos of the piezoelectric plate considering damage effects. The von Karman's plate theory is adopted to derive nonlinear dynamic equations of the piezoelectric plates with damage under a transverse periodic load. The Galerkin method and Runge-Kutta procedure are used to solve the nonlinear equations. The effect of damage value, damage position and electrical loads on the bifurcation and chaos of the piezoelectric plate are determined and discussed. Present results provide a theoretical basis for the design of dynamic stability and nondestructive testing of the piezoelectric structures.

Key Words: Piezoelectric plates, Tensor valued internal state variables, Bifurcation and chaos, Damage, Dynamic stability

1. Introduction

The use of piezoelectric materials has received considerable attention in intelligent structures due to the intrinsic direct and converse piezoelectric effects in recent years. The piezoelectric materials have been used as sensors or actuators for the control of the active shape or vibration of the structures. Defects such as micro cracks, voids, dislocations and delaminations are introduced in the piezoelectric materials during manufacturing and poling process. The existence of these defects such as localized flaw greatly affects the electric, dielectric, elastic, mechanical and piezoelectric properties of the piezoelectric materials, especially the global stability and dynamic bifurcation of piezoelectric structures. Therefore, it is important to understand the influence of these defects on the average mechanical properties of piezoelectric structures so that the stability predictions of the piezoelectric structures can be made.

Damage in fiber-reinforced composite materials has been vastly investigated, and many theories have been established and used to predict the life of composite structures. Based on the framework of irreversible thermodynamics with internal state variables, Talreja [1] developed a phenomenological theory for composite laminated plates. In his study, the Helmholtz free energy was expanded into polynomial in terms of elastic strains and damage variables to obtain the stiffness-damage relations. Utilizing a continuum mechanics approach, Allen [2, 3] developed a model for predicting the thermomechanical constitution of initially elastic composites subjected to both monotonic and cyclic fatigue loading. Valliappan [4] established the elastic constitutive equations for anisotropic damaged materials, and the implementation of these constitutive equations in the finite element analysis was explained. By defining damage variables as the material stiffness reduction, Ladeveze [5] formulated the constitutive equations and the corresponding damage evolution laws of the elementary ply for laminated composites that can be used to describe the matrix micro-cracking and fiber/matrix debonding. Schapery [6] discussed the homogenized constitutive equations for the mechanical behavior of unidirectional fiber composites with growing damage, and the emphasis was on resin matrices reinforced with high modulus elastic fiber. Zhang [7] investigated a computational model for the damage evolution of engineering materials under dynamic loading, and two models for dynamic damage evolution of materials in general anisotropic damage state were presented. Moore and Dillard [8] have observed time dependent growth of transverse cracks in graphite/epoxy and Kevlar/epoxy cross-ply laminated at room temperature. Luo and Daniel [9] have shown that the macroscopic mechanical behavior of unidirectional fiber-reinforced brittle matrix composites can be correlated explicitly with the microscopic deformation and damage.

Extensive studies have been made in stabilities of homogeneous plates. On the basis of the linear theory and the concept of the Lyapunov exponents, Aboudi [10] and Touati [11] investigated the dynamic stability of viscoelastic rectangular plate. Sun [12] studied the chaotic behaviors of viscoelastic plates subjected to an in-plane periodic load and pointed out that the stability of the structure can be increased by adjusting the material parameters. By employing Von Karman equations and Boltzmann superposition principle, Cheng [13, 14] revealed the dynamic properties of the viscoelastic rectangular plate with a transverse load in-plane periodic excitations. By applying the Leaderman nonlinear viscoelastic constitutive equation and linearization via output feedback, the control of the chaotic oscillations of viscoelastic plates was investigated by Chen [15]. Zhang [16] discussed the dynamic behavior of nonlinear viscoelastic Karman plates under a transverse harmonic load. Prabhakara [17, 18] analyzed the effect of the structural flaw on the natural frequency and buckling load of elastic plates subjected to a uniform in-plane load, and discussed the influence of different parameters on the linear static and dynamic behaviors of elastic plates with inhomogeneity subjected to non-uniform in-plane loads [19]. Laura [20] presented the linear fundamental frequency of transverse vibration for a damaged circular annular plate. However, up to now the analysis of bifurcation and chaos for the piezoelectric structures considering the damage influence under transverse periodic loads are not investigated and reported.

In present study, a new constitutive model for piezoelectric materials is presented by using the Talreja's tensor valued internal state damage variables and the Helmhotlz free energy. This model is applied to a specific case of analysis of bifurcation and chaos for piezoelectric plates under transverse periodic loads. By adopting von Karman's plate theory and using the Galerkin method, the nonlinear equations of motion for the piezoelectric plate with damage are derived and expressed in the term of time functions. In the numerical examples, the effect of damage value, damage position and electrical loads on the bifurcation and chaos of the piezoelectric plate are investigated and discussed.

2. Basic equations

2.1 Constitutive equations for damaged piezoelectric materials

Consider a representative volume element of a piezoelectric solid with a multitude of damage entities in the form of microcracks, as shown in Fig. 1. As discussed in Talreja, two vectors are needed to define each damage entity. These are damage influence vector $\vec{a}_i$ and unit normal $\vec{n}_i$ to damage entity surface. Damage influence vector represents an appropriately chosen effect of damage entity on the surrounding medium. With these two vectors, a damage entity tensor d_{ij} is formed by taking an integral of the diad $\vec{a}_i \cdot \vec{n}_i$ over the surface of the damage entity.

$$d_{ij} = \int_S \vec{a}_i \cdot \vec{n}_i dS \tag{1}$$

Now if there are n distinct damage modes in the representative volume element (e.g. intralaminar crack in different orientations etc,) denoted by $k = 1, 2, \ldots n$, a damage tensor can be defined for each mode as

$$\omega_{ij}^k = \frac{1}{V_r} \sum_{\vartheta_k} (d_{ij})_{\vartheta_k} \tag{2}$$

where V_r is the volume of the representative volume element and ϑ_k represents the number of damage entities in the kth damage mode. The tensor ω_{ij}^k is an unsymmetrical tensor in general. However, we can represent the vector $\vec{a}_i$ along the normal and tangential directions at any point on the surface of the damage entity and write

$$d_{ij} = d_{ij}^1 + d_{ij}^2 \tag{3}$$

where $d_{ij}^1 = \int_S a\vec{n}_i \cdot \vec{n}_j dS$ and $d_{ij}^2 = \int_S b\vec{m}_i \cdot \vec{n}_j dS$.

in which, a and b are the magnitudes of the normal and tangential projections of vector $\vec{a}_i$ respectively, and vectors $\vec{n}_i$ and $\vec{m}_i$ are unit normal and tangential vectors, respectively. Thus the damage tensor ω_{ij} can be written as

$$\omega_{ij}^k = \omega_{ij}^{1k} + \omega_{ij}^{2k} \tag{4}$$

where $\omega_{ij}^{1k} = \frac{1}{V_r}\sum_{\vartheta_k}(d_{ij}^1)_{\vartheta_k}$ and

$$\omega_{ij}^{2k} = \frac{1}{V_r}\sum_{\vartheta_k}(d_{ij}^2)_{\vartheta_k}$$

Physically, the damage tensor ω_{ij}^{1k} represents the effects of crack opening on the surrounding medium whereas the damage tensor ω_{ij}^{2k} represents the effects of sliding between the two crack faces. In many situations, the sliding between the crack faces can be negligible, e.g. for intralaminar cracks constrained by stiff plies, and hence we assume $\omega_{ij}^{2k} \equiv 0$. This implies $\omega_{ij}^k \equiv \omega_{ij}^{1k}$ which is a symmetric tensor.

For the case of damaged piezoelectric material where the damage is represented by internal state variables, the Helmholtz free energy of piezoelectric material can be written as a function of the transformed elastic strains, the electric field vector and damage internal variables, that is

$$H = H(\varepsilon_{ij}, E_i, \omega_{ij}^k) \tag{5}$$

The transformed stress components σ_{ij} and the electric displacement components D_i at any fixed damage state are now given by

$$\begin{aligned} \sigma_{ij} &= \frac{\partial H(\varepsilon_{ij}, E_i, \omega_{ij}^k)}{\partial \varepsilon_{ij}} \\ D_i &= -\frac{\partial H(\varepsilon_{ij}, E_i, \omega_{ij}^k)}{\partial E_i} \end{aligned} \tag{6}$$

When the damage induced by the cracks in matrix of the piezoelectric material has the orthotropic property, the irreducible integrity bases for a scalar polynomial function of two symmetric second rank tensors can be expressed as

$$\begin{gathered} \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, {\varepsilon_{23}}^2, {\varepsilon_{31}}^2, {\varepsilon_{12}}^2, \varepsilon_{12}\varepsilon_{23}\varepsilon_{31}, \\ \omega_{11}^k, \omega_{22}^k, \omega_{33}^k, (\omega_{23}^k)^2, (\omega_{31}^k)^2, (\omega_{12}^k)^2, \\ \omega_{12}^k\omega_{23}^k\omega_{31}^k, \\ \varepsilon_{23}\omega_{23}^k, \varepsilon_{31}\omega_{31}^k, \varepsilon_{12}\omega_{12}^k, \omega_{23}^k\varepsilon_{12}\varepsilon_{13}, \omega_{31}^k\varepsilon_{32}\varepsilon_{12}, \\ \omega_{12}^k\varepsilon_{13}\varepsilon_{23}, \\ \varepsilon_{23}\omega_{12}^k\omega_{13}^k, \varepsilon_{31}\omega_{32}^k\omega_{12}^k, \varepsilon_{12}\omega_{13}^k\omega_{23}^k \\ E_1, E_2, E_3 \\ (k = 1, 2, \cdots\cdots n) \end{gathered} \tag{7}$$

where n is the number of the cracks' direction in the material.

For a piezoelectric single-layer plate, the local coordinate system $o-123$ is selected, in which $1, 2$ denote the two principal direction of the piezoelectric plate, 3 is vertical to the midsurface. According to the Kirchhoff hypothesis for plate $\varepsilon_{13} = \varepsilon_{23} = 0$ and applying Voigt notation to describe strains and damage variables, the bases of invariants can be further written as

$$\begin{gathered} \varepsilon_1, \varepsilon_2, \varepsilon_3, {\varepsilon_6}^2, {\omega_1}^k, {\omega_2}^k, {\omega_3}^k, ({\omega_4}^k)^2, ({\omega_5}^k)^2, \\ ({\omega_6}^k)^2, {\omega_4}^k{\omega_5}^k{\omega_6}^k, \varepsilon_6{\omega_6}^k, \varepsilon_6{\omega_4}^k{\omega_5}^k \\ E_1, E_2, E_3 \\ (k = 1, 2, \cdots\cdots n) \end{gathered} \tag{8}$$

Using the above stated irreducible integrity bases, the Helmholtz free energy of piezoelectric materials can be expressed as a quadratic expression of the strains or the electric field intensity, a mixture quadratic expression of strains and electric field intensity and a linear expression of damage variables as follows

$$
\begin{aligned}
H = {} & C_1^0\varepsilon_1^2 + C_2^0\varepsilon_1\varepsilon_2 + C_3^0\varepsilon_2^2 + C_4^0\varepsilon_6^2 + C_5^0\varepsilon_1^2 + C_6^0\varepsilon_1\varepsilon_3 + C_7^0\varepsilon_2\varepsilon_3 - (\kappa_1^0 E_1^2 + \kappa_2^0 E_2^2 + \\
& \kappa_3^0 E_3^2 + \kappa_4^0 E_1 E_2 + \kappa_5^0 E_2 E_3 + \kappa_6^0 E_3 E_1) - (e_1^0 E_1\varepsilon_1 + e_2^0 E_1\varepsilon_2 + e_3^0 E_1\varepsilon_3 + e_4^0 E_2\varepsilon_1 + \\
& e_5^0 E_2\varepsilon_2 + e_6^0 E_2\varepsilon_3 + e_7^0 E_3\varepsilon_1 + e_8^0 E_3\varepsilon_2 + e_9^0 E_3\varepsilon_3) + \sum_{k=1}^{n}(C_1^k\varepsilon_1^2\omega_1^k + C_2^k\varepsilon_1^2\omega_2^k + C_3^k\varepsilon_1^2\omega_3^k \\
& + C_4^k\varepsilon_2^2\omega_1^k + C_5^k\varepsilon_2^2\omega_2^k + C_6^k\varepsilon_2^2\omega_3^k + C_7^k\varepsilon_3^2\omega_1^k + C_8^k\varepsilon_3^2\omega_2^k + C_9^k\varepsilon_3^2\omega_3^k + C_{10}^k\varepsilon_6^2\omega_1^k + \\
& C_{11}^k\varepsilon_6^2\omega_2^k + C_{12}^k\varepsilon_6^2\omega_3^k + C_{13}^k\varepsilon_1\varepsilon_2\omega_1^k + C_{14}^k\varepsilon_1\varepsilon_2\omega_2^k + C_{15}^k\varepsilon_1\varepsilon_2\omega_3^k + C_{16}^k\varepsilon_2\varepsilon_3\omega_1^k + C_{17}^k\varepsilon_2\varepsilon_3\omega_2^k \\
& + C_{18}^k\varepsilon_2\varepsilon_3\omega_3^k + C_{19}^k\varepsilon_3\varepsilon_1\omega_1^k + C_{20}^k\varepsilon_3\varepsilon_1\omega_2^k + C_{21}^k\varepsilon_3\varepsilon_1\omega_3^k + C_{22}^k\varepsilon_6\varepsilon_1\omega_6^k + C_{23}^k\varepsilon_6\varepsilon_2\omega_6^k + \\
& C_{24}^k\varepsilon_6\varepsilon_3\omega_6^k) - \sum_{k=1}^{n}(\kappa_1^k E_1^2\omega_1^k + \kappa_2^k E_1^2\omega_2^k + \kappa_3^k E_1^2\omega_3^k + \kappa_4^k E_2^2\omega_1^k + \kappa_5^k E_2^2\omega_2^k + \kappa_6^k E_3^2\omega_3^k + \\
& \kappa_7^k E_3^2\omega_1^k + \kappa_8^k E_3^2\omega_2^k + \kappa_9^k E_3^2\omega_3^k + \kappa_{10}^k E_1 E_2\omega_1^k + \kappa_{11}^k E_1 E_2\omega_2^k + \kappa_{12}^k E_1 E_2\omega_3^k + \kappa_{13}^k E_2 E_3\omega_1^k \\
& + \kappa_{14}^k E_2 E_3\omega_2^k + \kappa_{15}^k E_2 E_3\omega_3^k + \kappa_{16}^k E_3 E_1\omega_1^k + \kappa_{17}^k E_3 E_1\omega_2^k + \kappa_{18}^k E_3 E_1\omega_3^k) - \sum_{k=1}^{n}(e_1^k E_1\varepsilon_1\omega_1^k \\
& + e_2^k E_1\varepsilon_1\omega_2^k + e_3^k E_1\varepsilon_1\omega_3^k + e_4^k E_1\varepsilon_2\omega_1^k + e_5^k E_1\varepsilon_2\omega_2^k + e_6^k E_1\varepsilon_2\omega_3^k + e_7^k E_1\varepsilon_3\omega_1^k + e_8^k E_1\varepsilon_3\omega_2^k \\
& + e_9^k E_1\varepsilon_3\omega_3^k + e_{10}^k E_2\varepsilon_1\omega_1^k + e_{11}^k E_2\varepsilon_1\omega_2^k + e_{12}^k E_2\varepsilon_1\omega_3^k + e_{13}^k E_2\varepsilon_2\omega_1^k + e_{14}^k E_2\varepsilon_2\omega_2^k + e_{15}^k E_2\varepsilon_2\omega_3^k \\
& + e_{16}^k E_2\varepsilon_3\omega_1^k + e_{17}^k E_2\varepsilon_3\omega_2^k + e_{18}^k E_2\varepsilon_3\omega_3^k + e_{19}^k E_3\varepsilon_1\omega_1^k + e_{20}^k E_3\varepsilon_1\omega_2^k + e_{21}^k E_3\varepsilon_1\omega_3^k + e_{22}^k E_3\varepsilon_2\omega_1^k \\
& + e_{23}^k E_3\varepsilon_2\omega_2^k + e_{24}^k E_3\varepsilon_2\omega_3^k + e_{25}^k E_3\varepsilon_3\omega_1^k + e_{26}^k E_3\varepsilon_3\omega_2^k + e_{27}^k E_3\varepsilon_3\omega_3^k + e_{28}^k E_1\varepsilon_6\omega_6^k + e_{29}^k E_2\varepsilon_6\omega_6^k (\\
& + e_{30}^k E_3\varepsilon_6\omega_6^k) + P_0 + P_1\left(\varepsilon_p, \omega_q^k\right) + P_2\left(\omega_q^k\right) + P_3\left(E_i, \omega_q^k\right)
\end{aligned}
\tag{9}
$$

where $C_i^0 (i = 1,2,\text{L } 7)$ are the material constants without damage, $e_i^0 (i = 1,2,\text{L } 9)$ are the piezoelectric constants without damage, $\kappa_i^0 (i = 1,2,\text{L } 6)$ are the permittivity matrix constants without damage, $C_i^k (i = 1,2,\text{L } 24)$ are the material constants with damage; $e_i^k (i = 1,2,\text{L } 30)$ are the piezoelectric constants with damage, $\kappa_i^k (i = 1,2,\text{L } 18)$ are the permittivity matrix constants with damage, ρ is the density of piezoelectric material, P_0 is a constant, P_1 is a linear function of strains, P_2 is a linear function of damage variables and P_3 is a linear function of the electric field intensity. Then the stresses and the electric displacements can be expressed as

$$
\begin{aligned}
\sigma_p &= \frac{\partial H}{\partial \varepsilon_p} \\
&= [C_{pq}^0 + \sum_{k=1}^{n} C_{pq}^k]\varepsilon_q - [e_{pm}^0 + \sum_{k=1}^{n} e_{pm}^k]E_m \\
D_m &= -\frac{\partial H}{\partial E_m} \\
&= [e_{qm}^0 + \sum_{k=1}^{n} e_{qm}^k]^T \varepsilon_q + [\kappa_{mn}^0 + \sum_{k=1}^{n} \kappa_{mn}^k]E_n
\end{aligned}
\tag{10}
$$

in which C_{pq}^0, C_{pq}^k, κ_{mn}^0 and κ_{mn}^k are all symmetric matrixes having the forms as follows

$$
[C_{pq}^0] = \begin{bmatrix} 2C_1^0 & C_2^0 & C_6^0 & 0 \\ & 2C_3^0 & C_7^0 & 0 \\ & & 2C_5^0 & 0 \\ & & & 2C_4^0 \end{bmatrix}
\tag{11}
$$

$$[C_{pq}^k]=\begin{bmatrix} 2C_1^k\omega_1^k+2C_2^k\omega_2^k+2C_3^k\omega_3^k & C_{13}^k\omega_1^k+C_{14}^k\omega_2^k+C_{15}^k\omega_3^k & C_{19}^k\omega_1^k+C_{20}^k\omega_2^k+C_{21}^k\omega_3^k & C_{22}^k\omega_6^k \\ & 2C_4^k\omega_1^k+2C_5^k\omega_2^k+2C_6^k\omega_3^k & C_{16}^k\omega_1^k+C_{17}^k\omega_2^k+C_{18}^k\omega_3^k & C_{23}^k\omega_6^k \\ & & 2C_7^k\omega_1^k+2C_8^k\omega_2^k+2C_9^k\omega_3^k & C_{24}^k\omega_6^k \\ & & & 2C_{10}^k\omega_1^k+2C_{11}^k\omega_2^k+2C_{12}^k\omega_3^k \end{bmatrix} \tag{12}$$

$$[e_{pm}^0]=\begin{bmatrix} e_1^0 & e_4^0 & e_7^0 \\ e_2^0 & e_5^0 & e_8^0 \\ e_3^0 & e_6^0 & e_9^0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13}$$

$$[\kappa_{mn}^0]=\begin{bmatrix} 2\kappa_1^0 & \kappa_4^0 & \kappa_6^0 \\ & 2\kappa_2^0 & \kappa_5^0 \\ & & 2\kappa_3^0 \end{bmatrix} \tag{14}$$

$$[e_{pm}^k]=\begin{bmatrix} e_1^k\omega_1^k+e_2^k\omega_2^k+e_3^k\omega_3^k & e_{10}^k\omega_1^k+e_{11}^k\omega_2^k+e_{12}^k\omega_3^k & e_{19}^k\omega_1^k+e_{20}^k\omega_2^k+e_{21}^k\omega_3^k \\ e_4^k\omega_1^k+e_5^k\omega_2^k+e_6^k\omega_3^k & e_{13}^k\omega_1^k+e_{14}^k\omega_2^k+e_{15}^k\omega_3^k & e_{22}^k\omega_1^k+e_{23}^k\omega_2^k+e_{24}^k\omega_3^k \\ e_7^k\omega_1^k+e_8^k\omega_2^k+e_9^k\omega_3^k & e_{16}^k\omega_1^k+e_{17}^k\omega_2^k+e_{18}^k\omega_3^k & e_{25}^k\omega_1^k+e_{26}^k\omega_2^k+e_{27}^k\omega_3^k \\ e_{28}^k\omega_6^k & e_{29}^k\omega_6^k & e_{30}^k\omega_6^k \end{bmatrix} \tag{15}$$

$$[\kappa_{mn}^k]=\begin{bmatrix} 2\kappa_1^k\omega_1^k+2\kappa_2^k\omega_2^k+2\kappa_3^k\omega_3^k & \kappa_{10}^k\omega_1^k+\kappa_{11}^k\omega_2^k+\kappa_{12}^k\omega_3^k & \kappa_{16}^k\omega_1^k+\kappa_{17}^k\omega_2^k+\kappa_{18}^k\omega_3^k \\ & 2\kappa_4^k\omega_1^k+2\kappa_5^k\omega_2^k+2\kappa_6^k\omega_3^k & \kappa_{13}^k\omega_1^k+\kappa_{14}^k\omega_2^k+\kappa_{15}^k\omega_3^k \\ & & 2\kappa_7^k\omega_1^k+2\kappa_8^k\omega_2^k+2\kappa_9^k\omega_3^k \end{bmatrix} \tag{16}$$

Assuming that there is only a damage mode in the representative volume element, the relations of the strains, the stresses, the electric field intensity and the electric displacements in equation (10) can be simplified as

$$\begin{aligned} \sigma_p &= [C_{pq}^0+C_{pq}^1]\varepsilon_q-[e_{pm}^0+e_{pm}^1]E_m \\ D_m &= [e_{qm}^0+e_{qm}^1]^T\varepsilon_q+[\kappa_{mn}^0+\kappa_{mn}^1]E_n \end{aligned} \tag{17}$$

where C_{pq}^0, e_{pm}^0 and κ_{mn}^0 are the same as before. C_{pq}^k, e_{pm}^k, κ_{mn}^k are replaced by C_{pq}^1, e_{pm}^1, κ_{mn}^1.

In present study, consider that the matrix cracks in the piezoelectric plate are parallel to the coordinate plane $1-3$, all damage variables except ω_2 are zero, then the coefficient matrixes in equations (12), (15) and (16) can be simplified as

$$[C_{pq}^1]=\begin{bmatrix} 2C_2\omega_2 & C_{14}\omega_2 & C_{20}\omega_2 & 0 \\ & 2C_5\omega_2 & C_{17}\omega_2 & 0 \\ & & 2C_8\omega_2 & 0 \\ & & & 2C_{11}\omega_2 \end{bmatrix} \tag{18}$$

$$[e_{pm}^1]=\begin{bmatrix} e_2\omega_2 & e_{11}\omega_2 & e_{20}\omega_2 \\ e_5\omega_2 & e_{14}\omega_2 & e_{23}\omega_2 \\ e_8\omega_2 & e_{17}\omega_2 & e_{26}\omega_2 \\ 0 & 0 & 0 \end{bmatrix} \tag{19}$$

$$[\kappa_{mn}^1]=\begin{bmatrix} 2\kappa_2\omega_2 & \kappa_{11}\omega_2 & \kappa_{17}\omega_2 \\ & 2\kappa_5\omega_2 & \kappa_{14}\omega_2 \\ & & 2\kappa_8\omega_2 \end{bmatrix} \tag{20}$$

Due to cracks are parallel to the coordinate plane $1-3$, the effect of the damage on the plate stiffness in this coordinate plane $1-3$ can be neglected. Then matrix (18) can be further simplified as

$$[C_{pq}^1]=\begin{bmatrix} 0 & C_{14}\omega_2 & C_{20}\omega_2 & 0 \\ & 2C_5\omega_2 & C_{17}\omega_2 & 0 \\ & & 0 & 0 \\ & & & 2C_{11}\omega_2 \end{bmatrix} \tag{21}$$

Letting $\sigma_3=0$ and using equation (17), the constitutive relations with damage of the piezoelectric plate for the plane stress problems are obtained as follows

$$\begin{aligned} \sigma_p &= [C_{pq}]\varepsilon_q - [e_{pm}]E_m \\ &= [C^0_{pq} + C^1_{pq}]\varepsilon_q - [e^0_{pm} + e^1_{pm}]E_m \\ D_m &= [e_{qm}]^T \varepsilon_q + [\kappa_{mn}]E_n \\ &= [e^0_{qm} + e^1_{qm}]^T \varepsilon_q + [\kappa^0_{mn} + \kappa^1_{mn}]E_n \\ &(p,q = 1,2,6 \quad m,n = 1,2,3) \end{aligned} \tag{22}$$

where

$$[C^0_{pq}] = \begin{bmatrix} 2C^0_1 - \dfrac{(C^0_6)^2}{2C^0_5} & C^0_2 - \dfrac{C^0_6 C^0_7}{2C^0_5} & 0 \\ & 2C^0_3 - \dfrac{(C^0_7)^2}{2C^0_5} & 0 \\ & & 2C^0_4 \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} C^*_{11} & C^*_{12} & 0 \\ & C^*_{22} & 0 \\ & & C^*_{66} \end{bmatrix} \tag{23}$$

$$[C^1_{pq}] = \begin{bmatrix} \dfrac{C^0_6 C_{20}\omega_2}{C^0_5} & \left(C_{14} - \dfrac{C^0_7 C_{20} + C^0_6 C_{17}}{2C^0_5}\right)\omega_2 & 0 \\ & \left(2C_5 - \dfrac{C^0_7 C_{17}}{C^0_5}\right)\omega_2 & 0 \\ & & 2C_{11}\omega_2 \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ & \alpha_{22} & 0 \\ & & \alpha_{66} \end{bmatrix}\omega_2 \tag{24}$$

$$[e^0_{pm}] = \begin{bmatrix} e^0_1 - \dfrac{C^0_6 e^0_3}{2C^0_5} & e^0_4 - \dfrac{C^0_6 e^0_6}{2C^0_5} & e^0_7 - \dfrac{C^0_6 e^0_9}{2C^0_5} \\ e^0_2 - \dfrac{C^0_7 e^0_3}{2C^0_5} & e^0_5 - \dfrac{C^0_7 e^0_6}{2C^0_5} & e^0_8 - \dfrac{C^0_7 e^0_9}{2C^0_5} \\ 0 & 0 & 0 \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} e^*_{11} & e^*_{12} & e^*_{13} \\ e^*_{21} & e^*_{22} & e^*_{23} \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

$$[e^1_{pm}] = \begin{bmatrix} \left(e_2 - \dfrac{C^0_6 e_8 + C_{20} e^0_3}{2C^0_5}\right)\omega_2 & \left(e_{11} - \dfrac{C^0_6 e_{17} + C_{20} e^0_6}{2C^0_5}\right)\omega_2 & \left(e_{20} - \dfrac{C^0_6 e_{26} + C_{20} e^0_9}{2C^0_5}\right)\omega_2 \\ \left(e_5 - \dfrac{C^0_7 e_8 + C_{17} e^0_3}{2C^0_5}\right)\omega_2 & \left(e_{14} - \dfrac{C^0_6 e_{17} + C_{17} e^0_6}{2C^0_5}\right)\omega_2 & \left(e_{23} - \dfrac{C^0_7 e_{26} + C_{17} e^0_9}{2C^0_5}\right)\omega_2 \\ 0 & 0 & 0 \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ 0 & 0 & 0 \end{bmatrix}\omega_2 \tag{26}$$

$$[\kappa_{mn}^0]=\begin{bmatrix} 2\kappa_1^0+\dfrac{(e_3^0)^2}{2C_5^0} & \kappa_4^0+\dfrac{e_3^0e_6^0}{2C_5^0} & \kappa_6^0+\dfrac{e_3^0e_9^0}{2C_5^0} \\ & 2\kappa_2^0+\dfrac{(e_6^0)^2}{2C_5^0} & \kappa_5^0+\dfrac{e_6^0e_9^0}{2C_5^0} \\ & & 2\kappa_3^0+\dfrac{(e_9^0)^2}{2C_5^0} \end{bmatrix}\underline{\underline{def}}\begin{bmatrix} \kappa_{11}^{\bullet} & \kappa_{12}^{\bullet} & \kappa_{13}^{\bullet} \\ & \kappa_{22}^{\bullet} & \kappa_{23}^{\bullet} \\ & & \kappa_{33}^{\bullet} \end{bmatrix} \quad (27)$$

$$[\kappa_{mn}^1]=\begin{bmatrix} \left(2\kappa_2+\dfrac{e_3^0e_8}{C_5^0}\right)\omega_2 & \left(\kappa_{11}+\dfrac{e_3^0e_{17}+e_6^0e_8}{2C_5^0}\right)\omega_2 & \left(\kappa_{17}+\dfrac{e_3^0e_{26}+e_9^0e_8}{2C_5^0}\right)\omega_2 \\ & \left(2\kappa_5+\dfrac{e_6^0e_{17}}{C_5^0}\right)\omega_2 & \left(\kappa_{14}+\dfrac{e_6^0e_{26}+e_9^0e_{17}}{2C_5^0}\right)\omega_2 \\ & & \left(2\kappa_8+\dfrac{e_9^0e_{26}}{C_5^0}\right)\omega_2 \end{bmatrix}\underline{\underline{def}}\begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ & \gamma_{22} & \gamma_{23} \\ & & \gamma_{33} \end{bmatrix}\omega_2 \quad (28)$$

The relations between the electric fields E_x, E_y, E_z and the electric potential ϕ in the cartesian coordinate system are defined by

$$E_x=-\phi_{,x},\quad E_y=-\phi_{,y},\quad E_z=-\phi_{,z} \quad (29)$$

For the piezoelectric plate, only thickness direction electric field E_z is dominant. If the voltage applied to the piezoelectric plate with piezoelectric effect in the thickness only, then

$$E_z=V_e/h \quad (30)$$

where V_e is the applied voltage across the thickness of piezoelectric plates.

2.2. Basic equations of piezoelectric plates

Now, consider a piezoelectric plate with transverse cracks having thickness h, length a in the x-direction, width b in the y-direction subjected to a transverse periodic load $q(x,y,t)$ shown in Fig.2. A rectangular damaged region is bounded by $a_1\le x\le a_2$ and $b_1\le y\le b_2$. The centroid (c,d) of the damaged region represents the position of the damage. Defining S_d is a measure of the size of the damaged region. The reference surface defined by $z=0$ is set on the middle surface of the undeformed plate. The principle directions of the material are assumed to coincide with the coordinates x, y and z.

Letting u, v and w as the displacement components of an arbitrary point on the midsurface of the plate along the direction of x, y and z, respectively. According to classical nonlinear theory of plate, the strain components ε_x^0, ε_y^0 and γ_{xy}^0 of the midsurface can be written as

$$\begin{aligned} \varepsilon_x^0&=u_{,x}+\tfrac{1}{2}w_{,x}{}^2 \\ \varepsilon_y^0&=v_{,y}+\tfrac{1}{2}w_{,y}{}^2, \\ \gamma_{xy}^0&=u_{,y}+v_{,x}+w_{,x}w_{,y} \end{aligned} \quad (31)$$

and the curvatures κ_x, κ_y and κ_{xy} of midsurface as

$$\kappa_x=-w_{,xx},\quad \kappa_y=-w_{,yy},\quad \kappa_{xy}=-2w_{,xy} \quad (32)$$

then the nonlinear strain-displacement relations are expressed as follows

$$\begin{aligned} \varepsilon_x&=\varepsilon_x^0+z\kappa_x, \\ \varepsilon_y&=\varepsilon_y^0+z\kappa_y, \\ \gamma_{xy}&=\gamma_{xy}^0+z\kappa_{xy} \end{aligned} \quad (33)$$

Suppose the damage variable remains constant through the thickness of plate. Denote N_x, N_y, N_{xy} as the membrane stress resultants and M_x, M_y, M_{xy} as the stress couples of the plate. According to the classical nonlinear plate theory, the nonlinear governing equations of motion for the piezoelectric plate can be written as

$$\begin{aligned} & N_{x,x}+N_{xy,y}=0 \\ & N_{xy,x}+N_{y,y}=0 \\ & M_{x,xx}+2M_{xy,xy}+M_{y,yy}+N_x w_{,xx}+ \\ & +2N_{xy}w_{,xy}+N_y w_{,yy}+q(t)=\rho h w_{,tt} \end{aligned} \tag{34}$$

where ρ is the density of piezoelectric plate, and according to the classical theory of plate and using equations(22) and(33), the following constitutive equations can be obtained

$$\begin{aligned} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} &= \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} dz \\ &= \begin{bmatrix} A_{11}^* & A_{12}^* & 0 \\ & A_{22}^* & 0 \\ & & A_{66}^* \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - \begin{Bmatrix} N_x^p+\Delta N_x^p \\ N_y^p+\Delta N_y^p \\ N_{xy}^p+\Delta N_{xy}^p \end{Bmatrix} \end{aligned} \tag{35}$$

$$\begin{aligned} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} &= \int_{-h/2}^{h/2} z \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} dz \\ &= \begin{bmatrix} D_{11}^* & D_{12}^* & 0 \\ & D_{22}^* & 0 \\ & & D_{66}^* \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} M_x^p+\Delta M_x^p \\ M_y^p+\Delta M_y^p \\ M_{xy}^p+\Delta M_{xy}^p \end{Bmatrix} \end{aligned} \tag{36}$$

where N_x^p, N_x^p, N_{xy}^p and M_x^p, M_y^p, M_{xy}^p represent the component resultants and couples without damage due to the piezoelectric effect, respectively. $\Delta N_x^p, \Delta N_x^p, \Delta N_{xy}^p$ and $\Delta M_x^p, \Delta M_y^p, \Delta M_{xy}^p$ represent the decrement of the component resultants and couples with damage due to the piezoelectric effect. The resultants and couples due to the piezoelectric effect can be written as

$$\begin{aligned} & N_x^p = \int_{-h/2}^{h/2} e_{31}^0 \frac{V_e}{h} dz, \\ & N_y^p = \int_{-h/2}^{h/2} e_{32}^0 \frac{V_e}{h} dz, \ N_{xy}^p = 0, \\ & \Delta N_x^p = \int_{-h/2}^{h/2} e_{31}^1 \frac{V_e}{h} dz, \\ & \Delta N_y^p = \int_{-h/2}^{h/2} e_{32}^1 \frac{V_e}{h} dz, \ \Delta N_{xy}^p = 0, \\ & M_x^p = 0, M_y^p = 0, M_{xy}^p = 0, \\ & \Delta M_x^p = 0, \Delta M_y^p = 0, \Delta M_{xy}^p = 0 \end{aligned} \tag{37}$$

The stiffness coefficients A_{ij} and $D_{ij}(i,j=1,2,6)$ of the piezoelectric plate can be defined as follow

$$\begin{aligned} & A_{ij}^* = \int_{-h/2}^{h/2} (C_{ij}^0 + C_{ij}^1) dz = A_{ij} + \Delta A_{ij}, \\ & D_{ij}^* = \int_{-h/2}^{h/2} z^2 (C_{ij}^0 + C_{ij}^1) dz = D_{ij} + \Delta D_{ij} \\ & (i,j=1,2,6) \end{aligned} \tag{38}$$

where A_{ij} and D_{ij} represent the stiffness coefficients of the piezoelectric plate without damage respectively; ΔA_{ij} and ΔD_{ij} represent the decrement of the stiffness coefficients of the piezoelectric plate due to the damage effect respectively.

Introduce the following dimensionless parameters:

$$\xi = \frac{x}{a},\ \eta = \frac{y}{b},\ U = \frac{au}{h^2},\ V = \frac{av}{h^2},\ W = \frac{w}{h},$$

$$\lambda_1 = \frac{a}{b},\ \lambda_2 = \frac{a}{h},\ \bar{A}_{ij} = \frac{A_{ij}}{C_{11}^0 h}, \bar{D}_{ij} = \frac{D_{ij}}{C_{11}^0 h^3},$$

$$\Delta \bar{A}_{ij} = \frac{\Delta A_{ij}}{C_{11}^0 h}$$

$$\Delta \bar{D}_{ij} = \frac{\Delta D_{ij}}{C_{11}^0 h}, \bar{\alpha}_{ij} = \frac{\alpha_{ij}}{C_{11}^0},\ \bar{e}_{31} = \frac{e_{31}}{e_{31}^0},\ \bar{e}_{32} = \frac{e_{32}}{e_{31}^0},$$

$$\bar{\beta}_{31} = \frac{\beta_{31}}{e_{31}^0},\ \bar{\beta}_{32} = \frac{\beta_{32}}{e_{31}^0},\ \tau = \frac{t}{t_1},\ Q = \frac{q_t a^4}{C_{11}^0 h^4},$$

$$p = \frac{\rho a^4}{C_{11}^0 h^3 t_1^2},\ \bar{V}_e = \frac{e_{31}^0 V_e a^2}{C_{11}^0 h^3},$$

$$\varphi_1 = \frac{\bar{e}_{32}\bar{V}_e}{\bar{A}_{22}+\Delta\bar{A}_{22}},\ \varphi_2 = \frac{\bar{e}_{31}\bar{V}_e}{\bar{A}_{11}+\Delta\bar{A}_{11}} \quad (39)$$

By using equations (22), (34~39) and (42), the dimensionless nonlinear governing equations of piezoelectric plate with initial damage are obtained and expressed in terms of U, V and W as follows:

$$\begin{aligned}&(\bar{A}_{11}+\Delta\bar{A}_{11})(U_{,\xi\xi}+W_{,\xi}W_{,\xi\xi})+\\&+(\bar{A}_{12}+\Delta\bar{A}_{12})(\lambda_1 V_{,\xi\eta}+\lambda_1^2 W_{,\eta}W_{,\xi\eta})+\\&+(\bar{A}_{66}+\Delta\bar{A}_{66})(\lambda_1^2 U_{,\eta\eta}+\lambda_1 V_{,\xi\eta}\\&+\lambda_1^2 W_{,\eta}W_{,\xi\eta}+\lambda_1^2 W_{,\xi}W_{,\eta\eta})=0\end{aligned} \quad (40)$$

$$\begin{aligned}&(\bar{A}_{12}+\Delta\bar{A}_{12})(\lambda_1 U_{,\xi\eta}+\lambda_1 W_{,\xi}W_{,\xi\eta})+\\&+(\bar{A}_{22}+\Delta\bar{A}_{22})(\lambda_1^2 V_{,\eta\eta}+\lambda_1^3 W_{,\eta}W_{,\eta\eta})+\\&+(\bar{A}_{66}+\Delta\bar{A}_{66})(\lambda_1 U_{,\xi\eta}+V_{,\xi\xi}+\\&+\lambda_1 W_{,\eta}W_{,\xi\xi}+\lambda_1 W_{,\xi}W_{,\xi\eta})=0\end{aligned} \quad (41)$$

$$\begin{aligned}&-(\bar{D}_{11}+\Delta\bar{D}_{11})W_{,\xi\xi\xi\xi}-2\lambda_1^2(\bar{D}_{12}+\Delta\bar{D}_{12})W_{,\xi\xi\eta\eta}-\\&-4\lambda_1^2(\bar{D}_{66}+\Delta\bar{D}_{66})W_{,\xi\xi\eta\eta}-\lambda_1^4(\bar{D}_{22}+\Delta\bar{D}_{22})W_{,\eta\eta\eta\eta}+\\&+[(\bar{A}_{11}+\Delta\bar{A}_{11})(U_{,\xi}+\tfrac{1}{2}W_{,\xi}^2)+\\&+(\bar{A}_{12}+\Delta\bar{A}_{12})(\lambda_1 V_{,\eta}+\tfrac{1}{2}\lambda_1^2 W_{,\eta}^2)-\bar{e}_{31}\bar{V}_e]W_{,\xi\xi}+\\&+2(\bar{A}_{66}+\Delta\bar{A}_{66})(\lambda_1^2 U_{,\eta}+\lambda_1 V_{,\xi}+\lambda_1^2 W_{,\xi}W_{,\eta})W_{,\xi\eta}+\\&+[(\bar{A}_{12}+\Delta\bar{A}_{12})(\lambda_1^2 U_{,\xi}+\tfrac{1}{2}\lambda_1^2 W_{,\xi}^2)+\\&+(\bar{A}_{22}+\Delta\bar{A}_{22})(\lambda_1^3 V_{,\eta}+\tfrac{1}{2}\lambda_1^4 W_{,\eta}^2)-\lambda_1^2\lambda_2^2\bar{e}_{32}\bar{V}_e]W_{,\eta\eta}+\\&+Q-p\ddot{W}=0\end{aligned} \quad (42)$$

Suppose all the boundary of the plate is simply movable supported with zero boundary tangential displacement, the dimensionless boundary conditions can be written as

$$\begin{aligned}&\xi=0,1:\ V=W=N_\xi=M_\xi=0\\&\eta=0,1:\ U=W=N_\eta=M_\eta=0\end{aligned} \quad (43)$$

3. Solution methodology

A solution for equations (40~42) in conjunction with the boundary condition (43) is sought in the following separable form

$$\begin{aligned}&W=\sum_{m=1,3,\cdots}^{\infty}\sum_{n=1,3,\cdots}^{\infty}\bar{w}_{mn}(\tau)\sin m\pi\xi\sin n\pi\eta\\&U=-\frac{1}{2\pi}g(\eta)\sin 2\pi\xi+\\&+\sum_{m=1,3,\cdots}^{\infty}\sum_{n=1,3,\cdots}^{\infty}\bar{u}_{mn}(\tau)\cos m\pi\xi\sin n\pi\eta\\&V=-\frac{1}{2\pi}f(\xi)\sin 2\pi\eta+\\&+\sum_{m=1,3,\cdots}^{\infty}\sum_{n=1,3,\cdots}^{\infty}\bar{v}_{mn}(\tau)\sin m\pi\xi\cos n\pi\eta\end{aligned} \quad (44)$$

where the function $f(\xi)$ and $g(\eta)$ are defined by

$$f(\xi)=\frac{\lambda_1}{2}W_{,\eta}^2-\varphi_1,\quad \text{along } \eta=0 \text{ and } \eta=1,$$

$$g(\eta)=\frac{1}{2}W_{,\xi}^2-\varphi_2,\quad \text{along } \xi=0 \text{ and } \xi=1.$$

The transverse periodic load is assumed to be of the form

$$Q=F(\tau)\sin\pi\xi\sin\pi\eta,\ F(\tau)=F_0\sin\theta\tau \quad (45)$$

where F_0 is the external loading amplitude and θ is the exciting frequency.

Substituting equations (44) and (45) into equations (40~42)and multiplying them by $\cos i\pi\xi\sin j\pi\eta$, $\sin i\pi\xi\cos j\pi\eta$ and $\sin i\pi\xi\sin j\pi\eta$ respectively, then integrating them from 0 to 1 with respect to ξ and η , making use of the one-term approximation of the Galerkin method, the nonlinear differential equations that only includes time functions $\bar{u}(\tau), \bar{v}(\tau)$ and $\bar{w}(\tau)$ are obtained as.

$$\begin{aligned}&[-\frac{\pi^2}{4}(\bar{A}_{11}+\lambda_1^2\bar{A}_{66})+\Delta L_{11}]\bar{u}+\\&+[-\frac{\pi^2}{4}(\lambda_1\bar{A}_{12}+\lambda_1\bar{A}_{66})+\Delta L_{12}]\bar{v}+\\&+[\frac{4\pi}{9}(2\bar{A}_{11}2+2\lambda_1^2\bar{A}_{12}+\lambda_1^2\bar{A}_{66})+\\&+\Delta L_{13}]\bar{w}^2+[-\frac{16\varphi_2}{3\pi}\bar{A}_{11}+\Delta L_{14}]=0\end{aligned} \quad (46)$$

$$[-\frac{\pi^2}{4}(\lambda_1\bar{A}_{12}+\lambda_1\bar{A}_{66})+\Delta L_{21}]\bar{u}+$$
$$+[-\frac{\pi^2}{4}(\lambda_1^2\bar{A}_{22}+\bar{A}_{66})+\Delta L_{22}]\bar{v}+$$
$$+[\frac{4\pi}{9}(2\lambda_1\bar{A}_{12}+2\lambda_1^3\bar{A}_{22}+\lambda_1\bar{A}_{66})+$$
$$+\Delta L_{23}]\bar{w}^2+[-\frac{16\varphi_1}{3\pi}\lambda_1^2\bar{A}_{22}+\Delta L_{24}]=0 \quad (47)$$

$$[-\frac{\pi^4}{4}(\bar{D}_{11}+2\lambda_1^2\bar{D}_{12}+4\lambda_1^2\bar{D}_{66}+\lambda_1^4\bar{D}_{22})+$$
$$+\frac{\pi^2}{8}(\bar{A}_{11}\varphi_2+\lambda_1\bar{A}_{12}\varphi_1+\lambda_1^2\bar{A}_{12}\varphi_2+\lambda_1^2\bar{A}_{22}\varphi_1+$$
$$+2\bar{e}_{31}\bar{V}_e+2\lambda_1^2\lambda_2^2\bar{e}_{32}\bar{V}_e)+\Delta L_{31}]\bar{w}+$$
$$+[-\frac{\pi^4}{128}(9\bar{A}_{11}+9\lambda_1^4\bar{A}_{22}+4\lambda_1^2\bar{A}_{66}+18\lambda_1^2\bar{A}_{12})+$$
$$+\Delta L_{32}]\bar{w}^3+[\frac{8\pi}{9}(2\bar{A}_{11}$$
$$+\lambda_1^2\bar{A}_{66}+2\lambda_1^2\bar{A}_{12})+\Delta L_{33}]\overline{uw}+$$
$$+[\frac{8\pi}{9}(2\lambda_1\bar{A}_{12}+\lambda_1\bar{A}_{66}+2\lambda_1^3\bar{A}_{22})+\Delta L_{34}]\overline{vw}+$$
$$+\frac{1}{4}F=\frac{1}{4}p\ddot{\bar{w}} \quad (48)$$

where, the differential operators $\Delta L_{ij}(i=1\sim3, j=1\sim4)$ are given in the Appendix.

For brevity, equations (46~48) can be written as follows

$$M_{11}\bar{u}+M_{12}\bar{v}+M_{13}\bar{w}^2+M_{14}=0 \quad (49)$$
$$M_{21}\bar{u}+M_{22}\bar{v}+M_{23}\bar{w}^2+M_{24}=0 \quad (50)$$
$$M_{31}\bar{w}+M_{32}\bar{w}^3+M_{33}\overline{uw}+M_{34}\overline{vw}+F=p\ddot{\bar{w}} \quad (51)$$

where the differential operators $M_{ij}(i=1\sim3, j=1\sim4)$ are the coefficients of the corresponding equations (46)~ (48) and the expressions are given in the Appendix.

Solving equations (49) and (50), the $\bar{u}(\tau)$ and $\bar{v}(\tau)$ can be expressed by the $\bar{w}(\tau)$. Then, substituting the resultant expressions into equation (51) and introducing the linear damping item $\mu\dot{\bar{w}}(\tau)$, the following equation can be obtained

$$\phi_1\bar{w}+\phi_2\bar{w}^3+F-\mu\dot{\bar{w}}=p\ddot{\bar{w}} \quad (52)$$

where ϕ_1 and ϕ_2 are given in the Appendix.

Obviously, equation (52) is a nonlinear differential equation. Letting $y_1(\tau)=\bar{w}(\tau), y_2(\tau)=\dot{\bar{w}}(\tau)$, then equation (52) can be reduced to a set of nonlinear ordinary differential equations

$$\begin{aligned}\dot{y}_1&=y_2\\ \dot{y}_2&=p^{-1}(\phi_1y_1+\phi_2y_1^3-\mu y_2+F)\end{aligned} \quad (53)$$

The 4-order Runge-Kutta method is applied to solve the non-autonomous system (53).

4. Numerical results

In the following numerical calculations, consider the piezoelectric material is the ceramic PZT-5A and the material constants are taken as

$$E_{11}=E_{22}=61.0\text{GPa},\quad E_{33}=53.2\text{GPa},$$
$$\mu_{12}=0.35,\mu_{13}=\mu_{23}=0.38,\quad G_{12}=22.6\text{GPa},$$
$$G_{13}=G_{23}=21.1\text{GPa},\quad \rho=7750\,\text{kg/m}^3,$$
$$e_{31}=e_{31}=7.209\,\text{C/m}^2,\quad e_{33}=15.118\,\text{C/m}^2,$$
$$e_{24}=e_{15}=12.72\,\text{C/m}^2,$$
$$\kappa_{11}=\kappa_{22}=1.53\times10^{-8}\,\text{F/m},$$
$$\kappa_{33}=1.5\times10^{-8}\,\text{F/m} \quad (54)$$

In all examples, the geometric parameters are given as $\lambda_1=1, \lambda_2=20$. When the damage effect is in consideration, the material parameters related to damage in all examples are taken as $\bar{\alpha}_{11}=-0.5$, $\bar{\alpha}_{12}=-0.25$, $\bar{\alpha}_{22}=-0.5$, $\bar{\alpha}_{66}=-0.25$, $\bar{\beta}_{31}=-0.5$, $\bar{\beta}_{32}=-0.5$, $\mu=0.05$, $S_d=25\%$.

When the 4-order Runge-Kutta method is applied to solve the non-autonomous system (53), the corresponding initial values of the system are given as $\{y_{10}, y_{20}\}=\{0,1\}$.

Fig.3 shows the effect of damage value ω_2 on the bifurcation diagram of the deflection y_1 with the amplitude F_0 of the external load. The region are given as $\bar{V}_e = 0.001$ and $(c,d) = (0.5, 0.5)$, respectively. And the value of the damage is taken $\omega_2 = 0.2$ and $\omega_2 = 0.6$ respectively. It can be seen that the motion state of the system translates from one periodic motion to a double periodic bifurcation, then repeatedly from one periodic motion to a double periodic bifurcation, lastly into a chaotic motion state with the increase of the load. But the critical load to enter the chaotic state is different with the value of the damage. When the value of the damage ω_2 is 0.2, the system enters the chaotic motion at $F_0 = 53.4$, while for $\omega_2 = 0.6$, the system of enters the chaotic motion state at $F_0 = 42.5$, as shown in Fig.4. The Poincaré maps for the system present the fractal feature with similar cloud state and uncountable point set, which shows that the system has moved into the chaotic motion state. Hence, the increment of the damage value is adverse to the stability of the vibration for the piezoelectric plate.

Fig.5 shows the effect of the centroid of the damaged region on the bifurcation diagram of the deflection y_1 with the amplitude F_0 of the external load. The value of the damage and the electrical load are taken $\omega_2 = 0.6$ and $\bar{V}_e = 0.001$ respectively. And the centroid of the damaged region is taken as $(0.3, 0.3)$ and $(0.5, 0.5)$, respectively. From Fig.3(b) and Fig.5, it can be concluded that the centroid of the damaged region has rarely influence on the dynamic properties of the system when the amplitude of the load is relatively small, and the system is relatively stable. Similarly, the system changes from a double periodic bifurcation to a chaotic motion state with the increase of the load. At the same time, the reduction of the structure's stiffness will become bigger and the critical load to enter the chaotic motion state will reduce when the distance of the centroid of the damaged region to the mid-point of the plate decreases. When the centroid of the damaged region is $(0.3, 0.3)$, the system enters the chaotic motion state at $F_0 = 55.2$, while the centroid of the damaged region is $(0.5, 0.5)$, the critical load $F_0 = 53.4$, as shown in Fig.6 and Fig.4(b).

Fig.7 shows the effect of the electrical loads on the bifurcation diagram of the deflection under the amplitude of the external load $F_0 = 50$. The value of the damage and the centroid of the damaged region are taken as $\omega_2 = 0.6$ and $(0.5, 0.5)$, respectively. It can be illustrated from Fig.7, that the electric load has greatly influence on the dynamic properties of the system. Similarly, the system translates from a double periodic bifurcation to a chaotic motion with the increase of the electric loads. It can be concluded that the positive voltage can lead to the instability of the system while the negative voltage can keep the system stable. Fig.8 shows the Poincaré maps of the system at different electrical loads. It can be seen that when the electrical load is $\bar{V}_e = 0.011$, the system enters the chaotic motion state.

5. Conclusions

This paper presents a constitutive model for piezoelectric material based on the Talreja's tensor valued internal state damage variables and applied to the analysis of bifurcation and chaos of the piezoelectric plate with damage. Numerical results show that the damage and electrical loads both have obvious effects on the bifurcation and chaos behaviors of the piezoelectric structures. With the increase of the damage value or damaged region, the stability of the vibration for the piezoelectric structure will reduce. The positive voltage can lead to the instability of the system while the negative voltage can keep the system stable. Present research results will provide theoretical basis for the design of dynamic stability and nondestructive testing of the piezoelectric structures.

Acknowledgements: The support of this work from the National Natural Science Foundation of China (No.10572049) is greatly appreciated.

Appendix

1. $\Delta L_{ij}(i=1\sim3, j=1\sim4)$ in the equations (46~48)

$$\Delta L_{11} = \iint_S (\Delta\bar{A}_{11}\Pi_{1,\xi\xi} + \lambda_1^2\Delta\bar{A}_{66}\Pi_{1,\eta\eta})\Pi_1 d\xi d\eta$$

$$\Delta L_{12} = \iint_S (\lambda_1\Delta\bar{A}_{12} + \lambda_1\Delta\bar{A}_{66})\Pi_{2,\xi\eta}\Pi_1 d\xi d\eta$$

$$\Delta L_{13} = \iint_S (\Delta\bar{A}_{11}\Pi_{4,\xi\xi} + \lambda_1^2\Delta\bar{A}_{66}\Pi_{4,\eta\eta})\Pi_1 d\xi d\eta + \iint_S (\lambda_1^2\Delta\bar{A}_{12} + \lambda_1^2\Delta\bar{A}_{66})\Pi_{5,\xi\eta}\Pi_1 d\xi d\eta$$
$$+ \iint_S (\Delta\bar{A}_{11}\Pi_{3,\xi}\Pi_{3,\xi\xi} + \lambda_1^2\Delta\bar{A}_{12}\Pi_{3,\eta}\Pi_{3,\xi\eta} + \lambda_1^2\Delta\bar{A}_{66}\Pi_{3,\eta}\Pi_{3,\xi\eta} + \lambda_1^2\Delta\bar{A}_{66}\Pi_{3,\xi}\Pi_{3,\eta\eta})\Pi_1 d\xi d\eta$$

$$\Delta L_{14} = \iint_S \Delta\bar{A}_{11}\Pi_{6,\xi\xi}\Pi_1 d\xi d\eta$$

$$\Delta L_{21} = \iint_S (\lambda_1\Delta\bar{A}_{12} + \lambda_1\Delta\bar{A}_{66})\Pi_{1,\xi\eta}\Pi_2 d\xi d\eta$$

$$\Delta L_{22} = \iint_S (\lambda_1^2\Delta\bar{A}_{22}\Pi_{2,\eta\eta} + \Delta\bar{A}_{66}\Pi_{2,\xi\xi})\Pi_2 d\xi d\eta$$

$$\Delta L_{23} = \iint_S (\lambda_1\Delta\bar{A}_{12} + \lambda_1\Delta\bar{A}_{66})\Pi_{4,\xi\eta}\Pi_2 d\xi d\eta + \iint_S (\lambda_1^3\Delta\bar{A}_{22}\Pi_{5,\eta\eta} + \lambda_1\Delta\bar{A}_{66}\Pi_{5,\xi\xi})\Pi_2 d\xi d\eta$$
$$+ \iint_S (\lambda_1\Delta\bar{A}_{12}\Pi_{3,\xi}\Pi_{3,\xi\eta} + \lambda_1^3\Delta\bar{A}_{22}\Pi_{3,\eta}\Pi_{3,\eta\eta} + \lambda_1\Delta\bar{A}_{66}\Pi_{3,\eta}\Pi_{3,\xi\xi} + \lambda_1\Delta\bar{A}_{66}\Pi_{3,\xi}\Pi_{3,\xi\eta})\Pi_2 d\xi d\eta$$

$$\Delta L_{24} = \iint_S \lambda_1^2\Delta\bar{A}_{22}\Pi_{7,\eta\eta}\Pi_2 d\xi d\eta$$

$$\Delta L_{31} = \iint_S (-\Delta\bar{D}_{11}\Pi_{3,\xi\xi\xi\xi} - 2\lambda_1^2\Delta\bar{D}_{12}\Pi_{3,\xi\xi\eta\eta} - 4\lambda_1^2\Delta\bar{D}_{66}\Pi_{3,\xi\xi\eta\eta} - \lambda_1^4\Delta\bar{D}_{22}\Pi_{3,\eta\eta\eta\eta})\Pi_3 d\xi d\eta +$$
$$\iint_S (\Delta\bar{A}_{11}\Pi_{6,\xi}\Pi_{3,\xi\xi} + \lambda_1\Delta\bar{A}_{12}\Pi_{7,\eta}\Pi_{3,\xi\xi} + \lambda_1^2\Delta\bar{A}_{12}\Pi_{6,\xi}\Pi_{3,\eta\eta} + \lambda_1^3\Delta\bar{A}_{22}\Pi_{7,\eta}\Pi_{3,\eta\eta})\Pi_3 d\xi d\eta$$

$$\Delta L_{32} = \iint_S [(\Pi_{4,\xi}\Pi_{3,\xi\xi} + \tfrac{1}{2}\Pi_{3,\xi}\Pi_{3,\xi}\Pi_{3,\xi\xi})\Delta\bar{A}_{11} + (\lambda_1^2\Pi_{5,\eta}\Pi_{3,\xi\xi} + \tfrac{1}{2}\lambda_1^2\Pi_{3,\eta}\Pi_{3,\eta}\Pi_{3,\xi\xi} + \lambda_1^2\Pi_{4,\xi}\Pi_{3,\eta\eta}$$
$$+ \tfrac{1}{2}\lambda_1^2\Pi_{3,\xi}\Pi_{3,\xi}\Pi_{3,\eta\eta})\Delta\bar{A}_{12} + (2\lambda_1^2\Pi_{4,\eta}\Pi_{3,\xi\eta} + 2\lambda_1^2\Pi_{5,\xi}\Pi_{3,\xi\eta} + 2\lambda_1^2\Pi_{3,\xi}\Pi_{3,\eta}\Pi_{3,\xi\eta})\Delta\bar{A}_{66} +$$
$$(\lambda_1^4\Pi_{5,\eta}\Pi_{3,\eta\eta} + \tfrac{1}{2}\lambda_1^4\Pi_{3,\eta}\Pi_{3,\eta}\Pi_{3,\eta\eta})\Delta\bar{A}_{22}]\Pi_3 d\xi d\eta$$

$$\Delta L_{33} = \iint_S (\Delta\bar{A}_{11}\Pi_{1,\xi}\Pi_{3,\xi\xi} + 2\lambda_1^2\Delta\bar{A}_{66}\Pi_{1,\eta}\Pi_{3,\xi\eta} + \lambda_1^2\Delta\bar{A}_{12}\Pi_{1,\xi}\Pi_{3,\eta\eta})\Pi_3 d\xi d\eta$$

$$\Delta L_{34} = \iint_S (\lambda_1^2\Delta\bar{A}_{12}\Pi_{2,\eta}\Pi_{3,\xi\xi} + 2\lambda_1^2\Delta\bar{A}_{66}\Pi_{2,\xi}\Pi_{3,\xi\eta} + \lambda_1^4\Delta\bar{A}_{22}\Pi_{2,\eta}\Pi_{3,\eta\eta})\Pi_3 d\xi d\eta$$

where

$$\Pi_1 = \cos\pi\xi\sin\pi\eta,\ \Pi_2 = \sin\pi\xi\cos\pi\eta,\ \Pi_3 = \sin\pi\xi\sin\pi\eta$$

$$\Pi_4 = -\frac{\pi}{4}\sin 2\pi\xi\sin^2\pi\eta,\ \Pi_5 = -\frac{\pi}{4}\sin^2\pi\xi\sin 2\pi\eta$$

$$\Pi_6 = \frac{\varphi_2}{2\pi}\sin 2\pi\xi,\ \ \Pi_7 = \frac{\varphi_1}{2\pi}\sin 2\pi\eta,$$

2. $M_{ij}(i=1\sim3, j=1\sim4)$ in the equations (49~51)

$$M_{11} = -\frac{\pi^2}{4}(\lambda_1\bar{A}_{12} + \lambda_1\bar{A}_{66}) + \Delta L_{12},\ M_{12} = -\frac{\pi^2}{4}(\lambda_1\bar{A}_{12} + \lambda_1\bar{A}_{66}) + \Delta L_{12}$$

$$M_{13} = \frac{4\pi}{9}(2\bar{A}_{11}2 + 2\lambda_1^2\bar{A}_{12} + \lambda_1^2\bar{A}_{66}) + \Delta L_{13},\ M_{14} = -\frac{16\varphi_2}{3\pi}\bar{A}_{11} + \Delta L_{14}$$

$$M_{21}=-\frac{\pi^2}{4}(\lambda_1\bar{A}_{12}+\lambda_1\bar{A}_{66})+\Delta L_{21},\ M_{22}=-\frac{\pi^2}{4}(\lambda_1^2\bar{A}_{22}+\bar{A}_{66})+\Delta L_{22}$$

$$M_{23}=\frac{4\pi}{9}(2\lambda_1\bar{A}_{12}+2\lambda_1^3\bar{A}_{22}+\lambda_1\bar{A}_{66})+\Delta L_{23},\ M_{24}=-\frac{16\varphi_1}{3\pi}\lambda_1^2\bar{A}_{22}+\Delta L_{24}$$

$$M_{31}=-\pi^4(\bar{D}_{11}+2\lambda_1^2\bar{D}_{12}+4\lambda_1^2\bar{D}_{66}+\lambda_1^4\bar{D}_{22})+\frac{\pi^2}{2}(\bar{A}_{11}\varphi_2+\lambda_1\bar{A}_{12}\varphi_1+\lambda_1^2\bar{A}_{12}\varphi_2+\lambda_1^2\bar{A}_{22}\varphi_1$$

$$+2\bar{e}_{31}\bar{V}_e+2\lambda_1^2\lambda_2^2\bar{e}_{32}\bar{V}_e)+4\Delta L_{31},\ M_{32}=-\frac{\pi^4}{32}(9\bar{A}_{11}+9\lambda_1^4\bar{A}_{22}+4\lambda_1^2\bar{A}_{66}+18\lambda_1^2\bar{A}_{12})+4\Delta L_{32}$$

$$M_{33}=\frac{32\pi}{9}(2\bar{A}_{11}+\lambda_1^2\bar{A}_{66}+2\lambda_1^2\bar{A}_{12})+4\Delta L_{33},\ M_{14}=\frac{32\pi}{9}(2\lambda_1\bar{A}_{12}+\lambda_1\bar{A}_{66}+2\lambda_1^3\bar{A}_{22})+4\Delta L_{34}$$

3. ϕ_1 and ϕ_2 in the equation (52)

$$\phi_1=M_{31}-\frac{M_{14}M_{22}M_{33}-M_{12}M_{24}M_{33}}{M_{11}M_{22}-M_{12}M_{21}}-\frac{M_{14}M_{21}M_{34}-M_{11}M_{24}M_{34}}{M_{12}M_{21}-M_{11}M_{22}}$$

$$\phi_2=M_{34}-\frac{M_{13}M_{22}M_{33}-M_{12}M_{23}M_{33}}{M_{11}M_{22}-M_{12}M_{21}}-\frac{M_{13}M_{21}M_{34}-M_{11}M_{23}M_{34}}{M_{12}M_{21}-M_{11}M_{22}}$$

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