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Variational analysis of delamination growth for composite laminated cylindrical shells under circumferential concentrated load

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Abstract

The growth of delamination in cylindrical shells under external pressure may lead to structural failure. Based on the variational principle of moving boundary (Qian WC. Variational calculus and finite element. Beijing: Science Press; 1980 [in Chinese].) and considering the contact effect between delamination regions, in this paper, the nonlinear governing equations for the delaminated cylindrical shells are derived, and the corresponding boundary and matching conditions are given. Moreover, according to the Griffith criterion, the formulas of energy release rate along the delamination front are obtained. As the numerical example, the delamination growth of axisymmetrical laminated cylindrical shells is analyzed, and the effects of delamination sizes and depths, geometrical parameters, material properties and laminate stacking sequences on delamination growth are discussed.

Keywords: Composite; Laminated cylindrical shell; Delamination growth; Energy release rates; Contact effect

1. Introduction

Composite laminates are widely used in engineering because of their specific excellent properties, such as high strength-to-weight ratio, high stiffness-to-weight ratio and design flexibility, etc. But there will be delamination damage in composite laminates during the manufacturing processes, for instance, shocks in assembling procedures. Comparing with axial pressure, external pressure does not lead to local buckling in delaminated composite laminates, but it can drive delamination grow. And this will drastically weaken the stiffness and the load carrying capacity of the laminated structure, finally resulting in global buckling and failure of structure.

It is necessary to determine the stress fields of delamination front in order to analyze the delamination growth of laminated structures. But it is very difficult to analyze the stress of the delamination front due to its singularity. Whereas the energy release rate, which indicates intensity of stress fields along delamination front, is a finite value. As a result, most researches concerned with delamination growth are carried out from the aspect of energy release rate. At present, most of this research has been concerned with one-dimensional penetration delamination and two-dimensional inner delamination. For the former, the analysis is relatively simple because the energy release rate of each point along delamination front is identical. Chai et al. [1] first studied the buckling of film delamination in isotropic laminated plates by using beam-column theory and also discussed delamination growth by adopting the definition of energy release rate in fracture mechanics. Based on classical laminated theory, Sallam and Simitses [2] and Yin [3] examined the film delamination buckling and growth in isotropic and orthotropic laminated plates by using the J-integral. Applying the method of geometric nonlinear finite element and the virtual crack closure technique, Witcomb [4] calculated the energy release rate components and analyzed film delamination growth. Zafer and Fu [5] investigated one-dimensional delamination growth of laminated composites containing

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multiple delaminations. Two-dimensional inner delamination, which appears frequently in practice, is more difficult to be studied due to its complex growth mode along delamination front. Based on the assumption of delamination with self-similar growth and using Rayleigh-Ritz method, Chai [6] first analyzed film elliptical delamination of two-dimension and calculated the average energy release rate. Whitcomb and Shivakumar [7,8] analyzed the strain-energy release rate of quasi-isotropic plates by using the virtual crack closure technique. Based on the theory of fracture mechanics, Sheinman and Kardomateas [9] and Davidson et al. [10] analyzed energy release rates and stress intensity factors for delaminated composite laminates. Applying the finite element method, Nillsson et al. [11] researched delamination buckling and growth in a slender composite panel. Using first-order shear deformable theory, Park and Sanker [12] and Wang and Qiao [13] computed energy release rates of delaminated plates. Zhang and Yu [14] investigated the buckling and growth of pressed thin film delamination by using the high-order perturbation and shooting method. Jane et al. [15] discussed the postbuckling and growth of film rectangular delamination by adopting Rayleigh-Ritz method. Bottega [17] derived a general form of a growth law for arbitrary shaped delamination in layered plates by using the theorem of stationary potential energy coupled with moving boundary. Zhou and Fan [18] and Zhang and Yu [19] studied the film delamination growth of plate under compression by recourse to the moving boundary variational principle. The above studies are all about the analysis of delamination growth for beams and plates. Moreover, most of the studies are limited to discuss film delamination growth. Up to now, no investigation has been reported in field of delamination growth for cylindrical shells.

The essence of delamination growth is that the delamination boundary continually moves. Therefore, in present study, based on the variational principle of moving boundary [16] and considering the contact effect between delamination regions, the nonlinear governing equations of the delaminated cylindrical shells are derived, and the corresponding boundary and matching conditions are given. At the same time, the formula of energy release rate along the delamination front is obtained according to Griffith criterion. Then, by using finite difference method, the nonlinear governing equations are resolved. The obtained solutions are substituted into the formula of energy release rate and the value of energy release rate can be uniquely determined. In numerical examples, the effects of delamination sizes and depths, geometrical parameters, material properties and laminate stacking sequences on the delamination growth are discussed for axisymmetrical cylindrical shells and some significant conclusions are obtained as well.

2. Basic equations

Consider a cylindrical shell with throughout circumference delamination having midsurface radius R, thickness

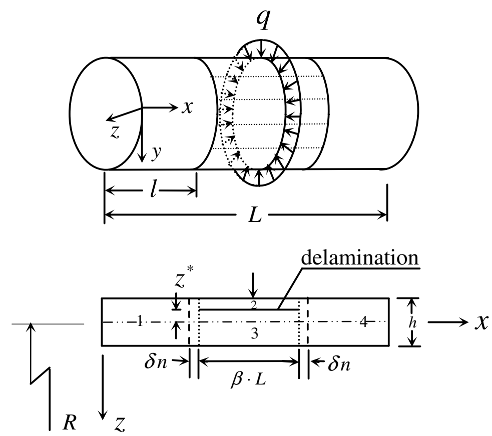


Fig. 1. Geometry of cylindrical shell with delamination.

h, length L and mass density ρ, and the shell is referred to the coordinate system x, y, z as shown in Fig. 1. The delaminated length of the shell is $\beta \cdot L$, and β is the delamination length parameter. z^* is the distance measured from the shell mid-surface to the delamination interface and l represents the delamination position measured from the left end of the shell. In order to investigate delamination growth, the delaminated cylindrical shell is divided into four regions which are denoted by Ω_i ($i = 1, 2, 3, 4$), respectively. Here, signs 2, 3 represent delaminated segments, and 1, 4 represent intact segments. The coordinate x for each region is measured from the left end. The thickness of regions 2 is h_2 and that of regions 3 is h_3, obviously, $h_2+h_3 = h$. In addition, the delamination growth for laminated cylindrical shell has two boundaries which are written as C_j ($j = 1, 2$). The boundaries on both ends of the shell are denoted by C_0.

Supposing that $\bar{u}_i, \bar{v}_i, \bar{w}_i$ denote the axial, circumferential and radial displacements of any points on the region Ω_i, respectively, and the corresponding displacement components of middle surface are u_i, v_i, w_i, respectively, then the displacement components are given by

$$\begin{aligned} \bar{u}_i(x,y,z) &= u_i(x,y) - z w_{i,x}(x,y) \\ \bar{v}_i(x,y,z) &= v_i(x,y) - z w_{i,y}(x,y) \\ \bar{w}_i(x,y,z) &= w_i(x,y) \end{aligned} \tag{1}$$

Assuming $\bar{\varepsilon}_{ix}, \bar{\varepsilon}_{iy}$ and $\bar{\varepsilon}_{ixy}$ denote the strain components of any points on region Ω_i, the nonlinear strain-displacement relations may be written as

$$\bar{\varepsilon}_{ix} = \varepsilon_{ix} + z\kappa_{ix}, \quad \bar{\varepsilon}_{iy} = \varepsilon_{iy} + z\kappa_{iy}, \quad \bar{\varepsilon}_{ixy} = \varepsilon_{ixy} + z\kappa_{ixy} \tag{2}$$

where ε_{ix}, ε_{iy}, ε_{ixy} are the strain components on the middle surface and κ_{ix}, κ_{iy}, κ_{ixy} are the change values of curvatures on the middle surface, and

$$\varepsilon_{ix} = u_{i,x} + \frac{1}{2}w_{i,x}^2, \quad \varepsilon_{iy} = v_{i,y} - \frac{w_i}{R_i} + \frac{1}{2}w_{i,y}^2$$

$$\varepsilon_{ixy} = u_{i,y} + v_{i,x} + w_{i,x}w_{i,y} \tag{3}$$

$$k_{ix} = -w_{i,xx}, k_{iy} = -w_{i,yy}, k_{ixy} = -2w_{i,xy} \tag{4}$$

According to classical theory of shells, the membrane stress resultants N_{ix}, N_{iy}, N_{ixy} and stress couples M_{ix}, M_{iy}, M_{ixy} can be written as

$$\begin{bmatrix}[N_i]\\ [M_i]\end{bmatrix} = \begin{bmatrix}\left[A_{ij}^{(i)}\right] & \left[B_{ij}^{(i)}\right]\\ \left[B_{ij}^{(i)}\right] & \left[D_{ij}^{(i)}\right]\end{bmatrix}\begin{Bmatrix}[\varepsilon_i]\\ [\kappa_i]\end{Bmatrix} \tag{5}$$

in which

$$[N_i] = \begin{Bmatrix}N_{ix}\\ N_{iy}\\ N_{ixy}\end{Bmatrix},\quad [M_i] = \begin{Bmatrix}M_{ix}\\ M_{iy}\\ M_{ixy}\end{Bmatrix},\quad [\varepsilon_i] = \begin{Bmatrix}\varepsilon_{ix}\\ \varepsilon_{iy}\\ \varepsilon_{ixy}\end{Bmatrix},$$
$$[\kappa_i] = \begin{Bmatrix}\kappa_{ix}\\ \kappa_{iy}\\ \kappa_{ixy}\end{Bmatrix} \tag{6}$$

$$\left(A_{ij}^{(i)}, B_{ij}^{(i)}, D_{ij}^{(i)}\right) = \int_{-h_i/2}^{h_i/2} \bar{Q}_{ij}^{(k)}\left(1, z, z^2\right)\mathrm{d}z \quad (i,j = 1,2,6) \tag{7}$$

where $A_{ij}^{(i)}, B_{ij}^{(i)}$ and $D_{ij}^{(i)}$ are the extension, coupling and bending rigidity, respectively, and $\bar{Q}_{ij}^{(k)}$ is elastic constant of the kth layer.

Assuming the circumferential concentrated load q acted on the middle of delaminated cylindrical shell, then the total potential energy of the delaminated cylindrical shell can be written as

$$\Pi = \sum_{i=1}^{4}\int\int\int_{\Omega_i} U_i\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \sum_{i=1}^{4}\int\int_{A_i} q w_i H(x)\,\mathrm{d}x\,\mathrm{d}y$$
$$-\sum_{i=1}^{4}\int\int_{A_i} T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}} w_i\,\mathrm{d}x\,\mathrm{d}y \tag{8}$$

where

$$H(x) = \begin{cases}1, & x = L/2\\ 0, & x \neq L/2\end{cases}$$

U_i is the strain energy density relative to region Ω_i. T_{c}^{i} is the coefficient of contact effect and $T_{\mathrm{c}}^{1} = T_{\mathrm{c}}^{4} = 0, T_{\mathrm{c}}^{2} = -1, T_{\mathrm{c}}^{3} = 1$. q^* is the contact force per unit length perpendicular to x-axis and it acts between region 2 and 3. R_{d} is the radius of curvature relative to delaminated interface.

The contact force q^* should satisfy the following condition:

$$q^* = \begin{cases}0 & \text{for } w_2 - w_3 \leqslant 0\\ f(w_2 - w_3) & \text{for } w_2 - w_3 > 0\end{cases} \tag{9}$$

The function $f(w_2 - w_3)$ can be chosen as a linear spring function, that is

$$f(w_2 - w_3) = k(w_2 - w_3) \tag{10}$$

where k is the elastic modulus. In this paper, k is approximated by an effective modulus of two springs connected in series. The two springs are namely the 2 and 3 regions beside the delamination. When a region, say, region 2, is subjected to a contact force q^*, the indentation can be approximated by

$$\delta_2 = \frac{q^* h_2}{E_2 2\pi R_{\mathrm{d}}} \tag{11}$$

As this force is also exerted on region 3 when the two regions are in contact, the indentation of region 3 is

$$\delta_3 = \frac{q^* h_3}{E_3 2\pi R_{\mathrm{d}}} \tag{12}$$

The relative displacement of the mid-surface of regions 2 and 3 is approximately

$$w_2 - w_3 = \frac{1}{2}(\delta_2 + \delta_3) \tag{13}$$

Substituting Eqs. (11) and (12) into Eq. (13) and according to Eq. (10), the spring modulus k is

$$k = \frac{4\pi R_{\mathrm{d}}}{h_2/E_2 + h_3/E_3} \tag{14}$$

Because w_2 and w_3 are unknown parameters to be solved, it is impossible to determine which branch of the q^* in Eq. (9) is to be used. To circumvent this difficulty, we will construct a function which will approximate q^* in Eq. (9) for any value of $w_2 - w_3$ to any desired degree of accuracy. It is noted that for any value of $w_2 - w_3$, q^* in Eq. (9) can be expressed as

$$q^* = \max[f_1(w_2 - w_3), f_2(w_2 - w_3)] \tag{15}$$

where

$$f_1(w_2 - w_3) = 0$$
$$f_2(w_2 - w_3) = k(w_2 - w_3)$$

The q^* in Eq. (15) can be approximated by the following expression

$$\hat{q} = \alpha(f_1, f_2)f_1 + \beta(f_1, f_2)f_2 \tag{16}$$

where $\alpha(f_1,f_2)$ and $\beta(f_1,f_2)$ are functions of f_1 and f_2, and they should satisfy the following conditions:

$$\begin{aligned}&\text{as } f_1 > f_2, \quad \alpha \to 1,\ \beta \to 0\\ &\text{as } f_1 \leqslant f_2, \quad \alpha \to 0,\ \beta \to 1\end{aligned} \tag{17}$$

$\alpha(f_1,f_2)$ and $\beta(f_1,f_2)$ satisfying the above conditions can be chosen as

$$\begin{aligned}\alpha(f_1,f_2) &= \tfrac{1}{2}[1 + \tanh A(f_1 - f_2)]\\ \beta(f_1,f_2) &= \tfrac{1}{2}[1 - \tanh A(f_1 - f_2)]\end{aligned} \tag{18}$$

where the parameter A is an artificially chosen large number depending upon the desired accuracy of approximation (A is taken as 10^{15} in present study).

As $f_1(w_2 - w_3) = 0$, the contact force q^* can be approximated by

$$\hat{q} = \frac{1}{2}[1 + \tan h(Af_2)]f_2 \tag{19}$$

In the above equation, letting f_2 be substituted with $k(w_2 - w_3)$, then we have

$$\hat{q} = \frac{1}{2}[1 + \tanh\,(Ak(w_2 - w_3))]k(w_2 - w_3) \tag{20}$$

Eq. (20) is the formula of calculating contact force between delaminations.

Using Eq. (5) and noticing that the problem of delamination growth is the variation of moving boundary, then from Eq. (8), the variation of the total potential energy is

$$\delta\Pi=\sum_{i=1}^{4}\int\int_{A_i+\delta A_i}\frac{1}{2}\begin{Bmatrix}[\varepsilon^{(i)}+\delta\varepsilon^{(i)}]\\ [\kappa^{(i)}+\delta\kappa^{(i)}]\end{Bmatrix}^{\mathrm{T}}\times\begin{bmatrix}[A^{(i)}] & [B^{(i)}]\\ [B^{(i)}] & [D^{(i)}]\end{bmatrix}\begin{Bmatrix}[\varepsilon^{(i)}+\delta\varepsilon^{(i)}]\\ [\kappa^{(i)}+\delta\kappa^{(i)}]\end{Bmatrix}\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{4}\int\int_{A_i+\delta A_i}q(w_i+\delta w_i)H(x)\,\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{4}\int\int_{A_i+\delta A_i}T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}(w_i+\delta w_i)\,\mathrm{d}x\,\mathrm{d}y-\Pi \quad (21)$$

Moreover, it is assumed that the shell is symmetrically laminated and all regions are still symmetric with respect to their each midsurface after delamination occurs. Then, $[B^{(\mathrm{i})}]=0$ and the above equation can be written as

$$\delta\Pi=\sum_{i=1}^{4}\int\int_{A_i}\left\{[\varepsilon^{(i)}]^{\mathrm{T}}[A^{(i)}][\delta\varepsilon^{(i)}]+[\kappa^{(i)}]^{\mathrm{T}}[D^{(i)}][\delta\kappa^{(i)}]\right\}\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{4}\int\int_{A_i}qH(x)\delta w_i\,\mathrm{d}x\mathrm{d}y-\sum_{i=1}^{4}\int\int_{A_i}T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}\delta w_i\,\mathrm{d}x\mathrm{d}y+\frac{1}{2}\sum_{i=1}^{4}\int\int_{\delta A_i}\left\{[\varepsilon^{(i)}]^{\mathrm{T}}[A^{(i)}][\varepsilon^{(i)}]+[\kappa^{(i)}]^{\mathrm{T}}[D^{(i)}][\kappa^{(i)}]\right\}\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{4}\int\int_{\delta A_i}qH(x)w_i\,\mathrm{d}x\mathrm{d}y-\sum_{i=1}^{4}\int\int_{\delta A_i}T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}w_i\,\mathrm{d}x\mathrm{d}y \quad (22)$$

The last three items of Eq. (22) can be given in the following form:

$$\frac{1}{2}\sum_{i=1}^{4}\int\int_{\delta A_i}\left\{[\varepsilon^{(i)}]^{\mathrm{T}}[A^{(i)}][\varepsilon^{(i)}]+[\kappa^{(i)}]^{\mathrm{T}}[D^{(i)}][\kappa^{(i)}]\right\}\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{4}\int\int_{\delta A_i}qH(x)w_i\,\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{4}\int\int_{\delta A_i}T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}w_i\,\mathrm{d}x\,\mathrm{d}y=\sum_{i=1}^{4}\oint_{C_j}\left\{\frac{1}{2}\left([\varepsilon^{(i)}]^{\mathrm{T}}[A^{(i)}][\varepsilon^{(i)}]+[\kappa^{(i)}]^{\mathrm{T}}[D^{(i)}][\kappa^{(i)}]\right)-qH(x)w_i-T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}w_i\right\}\delta n_i\,\mathrm{d}C_j \quad (j=1,2) \quad (23)$$

Using Eqs. (3)–(6) and (23), Eq. (22) can be changed to

$$\delta\Pi=\sum_{i=1}^{4}\int\int_{A_i}\left\{N_{xi}\delta\left(u_{i,x}+\frac{1}{2}w_{i,x}^2\right)+N_{yi}\delta\left(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2\right)+N_{xyi}\delta(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})+M_{xi}\delta(-w_{i,xx})+M_{yi}\delta(-w_{i,yy})+M_{xyi}\delta(-2w_{i,xy})\right\}\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{4}\int\int_{A_i}qH(x)\delta w_i\,\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{4}\int\int_{A_i}T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}\delta w_i\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2}\sum_{i=1}^{4}\oint_{C_j}\left\{N_{xi}\left(u_{i,x}+\frac{1}{2}w_{i,x}^2\right)+N_{yi}\left(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2\right)+N_{xyi}(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})+M_{xi}(-w_{i,xx})+M_{yi}(-w_{i,yy})+M_{xyi}(-2w_{i,xy})-qH(x)w_i-T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}w_i\right\}\delta n_i\,\mathrm{d}C_j \quad (24)$$

Further processing the above equation, it can be written as the following two parts, that is

$$\delta\Pi=\delta\Pi_1+\delta\Pi_2 \quad (25)$$

where

$$\delta\Pi_1=\sum_{i=1}^{4}\int\int_{A_i}\left\{\left[-\frac{\partial}{\partial x}(N_{xi})-\frac{\partial}{\partial y}(N_{xyi})\right]\delta u_i+\left[-\frac{\partial}{\partial x}(N_{xyi})-\frac{\partial}{\partial y}(N_{yi})\right]\delta v_i+\left[-\frac{\partial}{\partial x}(M_{xi,x})-\frac{\partial}{\partial y}(M_{yi,y})-\frac{\partial}{\partial y}(M_{xyi,x})-\frac{\partial}{\partial x}(M_{xyi,y})-\frac{\partial}{\partial x}(N_{xi}w_{i,x})-\frac{\partial}{\partial y}(N_{yi}w_{i,y})-\frac{\partial}{\partial x}(N_{xyi}w_{i,y})-\frac{\partial}{\partial y}(N_{xyi}w_{i,x})-\frac{N_{yi}}{R_i}-qH(x)-T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}\right]\delta w_i\right\}\mathrm{d}x\mathrm{d}y+\sum_{i=1}^{4}\oint_{C_j}\left\{N_{xi}\delta(u_i|_{C_j})+N_{xyi}\delta(v_i|_{C_j})+(M_{xi,x}+M_{xyi,y}+N_{xi}w_{i,x}+N_{xyi}w_{i,y})\delta(w_i|_{C_j})-M_{xi}\delta(w_{i,x}|_{C_j})-M_{xyi}\delta(w_{i,y}|_{C_j})\right\}\mathrm{d}C_j \quad (26)$$

$$\delta\Pi_2=\sum_{i=1}^{4}\oint_{C_j}\left[N_{xi}\left(-\frac{\partial u_i}{\partial n}\bigg|_{C_j}\right)+N_{xyi}\left(-\frac{\partial v_i}{\partial n}\bigg|_{C_j}\right)+(M_{xi,x}+M_{xyi,y}+N_{xi}w_{i,x}+N_{xyi}w_{i,y})\left(-\frac{\partial w_i}{\partial n}\bigg|_{C_j}\right)+M_{xi}\frac{\partial w_{i,x}}{\partial n}\bigg|_{C_j}+M_{xyi}\frac{\partial w_{i,y}}{\partial n}\bigg|_{C_j}\right]\cdot\delta n_i\,\mathrm{d}C_j+\frac{1}{2}\sum_{i=1}^{4}\oint_{C_j}\left\{N_{xi}\left(u_{i,x}+\frac{1}{2}w_{i,x}^2\right)+N_{yi}\left(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2\right)+N_{xyi}(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})+M_{xi}(-w_{i,xx})+M_{yi}(-w_{i,yy})+M_{xyi}(-2w_{i,xy})-qH(x)w_i-T_{\mathrm{c}}^{i}(q^*/2\pi R_{\mathrm{d}})w_i\right\}\delta n_i\,\mathrm{d}C_j \quad (27)$$

The derivations of Eq. (25) are given in Appendix A.

For the delaminated cylindrical shell, the normal direction n of delamination growth is consistent with the axial direction x. Therefore, Eq. (27) can also be written as follows:

$$\delta\Pi_2=\sum_{i=1}^{4}\oint_{C_j}[N_{xi}(-u_{i,x})+N_{xyi}(-v_{i,x})+(M_{xi,x}+M_{xyi,y}+N_{xi}w_{i,x}+N_{xyi}w_{i,y})(-w_{i,x})$$

$$+M_{xi}w_{i,xx}+M_{xyi}w_{i,xy}]\cdot\delta n_i\,\mathrm{d}C_j$$
$$+\frac{1}{2}\sum_{i=1}^{4}\oint_{C_j}\left\{N_{xi}\left(u_{i,x}+\frac{1}{2}w_{i,x}^2\right)+N_{yi}\left(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2\right)\right.$$
$$+N_{xyi}(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})+M_{xi}(-w_{i,xx})$$
$$+M_{yi}(-w_{i,yy})+M_{xyi}(-2w_{i,xy})-qH(x)w_i$$
$$\left.-T_{\mathrm{c}}^{i}(q^*/2\pi R_{\mathrm{d}})w_i\right\}\delta n_i\,\mathrm{d}C_j \tag{28}$$

Let

$$G_i=[N_{xi}(-u_{i,x})+N_{xyi}(-v_{i,x})+(M_{xi,x}+M_{xyi,y}$$
$$+N_{xi}w_{i,x}+N_{xyi}w_{i,y})(-w_{i,x})+M_{xi}w_{i,xx}+M_{xyi}w_{i,xy}]$$
$$+\frac{1}{2}[N_{xi}(u_{i,x}+\frac{1}{2}w_{i,x}^2)+N_{yi}(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2)$$
$$+N_{xyi}(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})+M_{xi}(-w_{i,xx})$$
$$+M_{yi}(-w_{i,yy})+M_{xyi}(-2w_{i,xy})]-qH(x)w_i$$
$$-T_{\mathrm{c}}^{i}(q^*/2\pi R_{\mathrm{d}})w_i \tag{29}$$

then

$$\delta\Pi_2=\sum_{i=1}^{4}\oint_{C_j}G_i\delta n_i\,\mathrm{d}C_j \tag{30}$$

The displacements of laminated cylindrical shell must change after imaginary growth δn occurs along delamination front. At the same time, the changeable area of the integral region is that $\delta A_i=\int_{C_j}\delta n_i\,\mathrm{d}C_j$. Thus, $\delta\Pi_2$ is the variation of potential energy due to the area alteration of each region. Because $\delta\Pi_1$ is the variation of potential energy due to the virtual displacement of laminated cylindrical shell while imaginary growth does not occur (i.e. the delamination boundary is immovable), so according to the principle of virtual displacement, when the laminated cylindrical shell is in state of equilibrium, we have

$$\delta\Pi_1=0 \tag{31}$$

From Eq. (31), the nonlinear governing equations, the corresponding boundary and matching conditions can be derived for the delaminated cylindrical shell.

The nonlinear governing equations for each region are

$$A_{11}^{(i)}u_{i,xx}+(A_{12}^{(i)}+A_{66}^{(i)})v_{i,xy}+(A_{12}^{(i)}+A_{66}^{(i)})w_{i,y}w_{i,xy}$$
$$+A_{11}^{(i)}w_{i,x}w_{i,xx}+A_{66}^{(i)}w_{i,x}w_{i,yy}+A_{66}^{(i)}u_{i,yy}-A_{12}^{(i)}\frac{1}{R_i}w_{i,x}=0$$
$$(A_{12}^{(i)}+A_{66}^{(i)})u_{i,xy}+A_{66}^{(i)}v_{i,xx}+A_{22}^{(i)}v_{i,yy}+(A_{12}^{(i)}+A_{66}^{(i)})w_{i,x}w_{i,xy}$$
$$+A_{66}^{(i)}w_{i,y}w_{i,xx}+A_{22}^{(i)}w_{i,y}w_{i,yy}-A_{22}^{(i)}\frac{1}{R_i}w_{i,y}=0$$
$$-(D_{11}^{(i)}w_{i,xxxx}+(2D_{12}^{(i)}+4D_{66}^{(i)})w_{i,xxyy}+D_{22}^{(i)}w_{i,yyyy})$$
$$+w_{i,xx}(A_{11}^{(i)}u_{i,x}+\frac{1}{2}A_{11}^{(i)}w_{i,x}^{2}+A_{12}^{(i)}v_{i,y}-A_{12}^{(i)}\frac{1}{R_i}w_i$$
$$+\frac{1}{2}A_{12}^{(i)}w_{i,y}^{2})+2A_{66}^{(i)}(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})w_{i,xy}$$
$$+(A_{12}^{(i)}u_{i,x}+\frac{1}{2}A_{12}^{(i)}w_{i,x}^{2}+A_{22}^{(i)}v_{i,y}-A_{22}^{(i)}\frac{1}{R_i}w_i+\frac{1}{2}A_{22}^{(i)}w_{i,y}^{2})$$
$$\times(w_{i,yy}+\frac{1}{R_i})+qH(x)+T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}=0 \tag{32}$$

The continuity conditions of displacements are

$$\begin{aligned}
u_2(0,y)&=u_1(L_1,y)+d_2w_{1,x}(L_1,y),\\
v_2(0,y)&=v_1(L_1,y)+d_2w_{1,y}(L_1,y)\\
u_3(0,y)&=u_1(L_1,y)-d_3w_{1,x}(L_1,y),\\
v_3(0,y)&=v_1(L_1,y)-d_3w_{1,y}(L_1,y)\\
u_2(L_2,y)&=u_4(0,y)+d_2w_{4,x}(0,y),\\
v_2(L_2,y)&=v_4(0,y)+d_2w_{4,y}(0,y)\\
u_3(L_3,y)&=u_4(0,y)-d_3w_{4,x}(0,y),\\
v_3(L_3,y)&=v_4(0,y)-d_3w_{4,y}(0,y)\\
w_1(L_1,y)&=w_2(0,y),w_1(L_1,y)=w_3(0,y),w_{1,x}(L_1,y)\\
&=w_{2,x}(0,y),w_{1,x}(L_1,y)=w_{3,x}(0,y)\\
w_4(0,y)&=w_2(L_2,y),w_4(0,y)=w_3(L_3,y),w_{4,x}(0,y)\\
&=w_{2,x}(L_2,y),w_{4,x}(0,y)=w_{3,x}(L_3,y)
\end{aligned} \tag{33}$$

The equilibrium conditions of moments and forces are

$$\begin{aligned}
N_{1xx}(L_1,y)&=N_{2xx}(0,y)+N_{3xx}(0,y),\\
N_{4xx}(0,y)&=N_{2xx}(L_2,y)+N_{3xx}(L_3,y)\\
N_{1xy}(L_1,y)&=N_{2xy}(0,y)+N_{3xy}(0,y),\\
N_{4xy}(0,y)&=N_{2xy}(L_2,y)+N_{3xy}(L_3,y)\\
M_{1x}(L_1,y)&=M_{2x}(0,y)-d_2N_{2_{xx}}(0,y)\\
&\quad+M_{3x}(0,y)+d_3N_{3_{xx}}(0,y),\\
M_{4x}(0,y)&=M_{2x}(L_2,y)-d_2N_{2_{xx}}(L_2,y)\\
&\quad+M_{3x}(L_3,y)+d_3N_{3_{xx}}(L_3,y)\\
Q_{1x}(L_1,y)&=Q_{2x}(0,y)-d_2N_{2_{xy,y}}(0,y)\\
&\quad+Q_{3x}(0,y)+d_3N_{3_{xy,y}}(0,y),\\
Q_{4x}(0,y)&=Q_{2x}(L_2,y)-d_2N_{2_{xy,y}}(L_2,y)\\
&\quad+Q_{3x}(L_3,y)+d_3N_{3_{xy,y}}(L_3,y)
\end{aligned} \tag{34}$$

where $Q_{\mathrm{ix}}=M_{\mathrm{ix,x}}+2M_{\mathrm{ixy,y}}$.

The boundary conditions for both ends are

$$\begin{aligned}
&w_1(0,y)=0,\quad N_{1xx}(0,y)=0,\quad N_{1xy}(0,y)=0,\quad w_{1,x}(0,y)=0\\
&w_4(L_4,y)=0,\quad N_{4xx}(L_4,y)=0,\quad N_{4xy}(L_4,y)=0,\quad w_{4,x}(L_4,y)=0\\
&\text{(clamped)}
\end{aligned} \tag{35}$$

$$\begin{aligned}
&w_1(0,y)=0,\quad N_{1xx}(0,y)=0,\quad N_{1xy}(0,y)=0,\quad w_{1,xx}(0,y)=0\\
&w_4(L_4,y)=0,\quad N_{4xx}(L_4,y)=0,\quad N_{4xy}(L_4,y)=0,\quad w_{4,xx}(L_4,y)=0\\
&\text{(simply supported)}
\end{aligned} \tag{36}$$

where $d_i=h/2-h_i/2$.

Altogether, when the imaginary growth δn occurs, the variation of total potential energy is (Noticing $\delta\Pi_1=0$)

$$\delta\Pi=\delta\Pi_2=\sum_{i=1}^{4}\oint_{C_j}G_i\delta n_i\,\mathrm{d}C_j \tag{37}$$

3. Energy release rate

The essence of delamination growth is that the delamination boundary continually moves and so the formulas of energy release rate can be founded according to Griffith criterion of crack growth. Assuming when imaginary growth δn occurs, the area change of delamination region is δA, base on energy conservation principle, the work done by the external loads is summation of the elastic strain energy and the energy spent on the delamination growth, that is

$$\delta(\Pi + \Gamma) = 0 \tag{38}$$

where $\Pi = U - W$, and the Π represents the total potential energy of elastic system, U represents the strain energy, W represents the work done by the external loads and Γ represents the energy spent on the delamination growth. According to Griffith criterion, the energy release rate G can be expressed as

$$G = -\lim_{\delta A \to 0} \frac{\delta \Pi}{\delta A} \tag{39}$$

So the average energy release rate G_a of delamination growth is

$$G_a = -\frac{\delta \Pi}{\delta A} = -\frac{\sum_{i=1}^{4} \oint_{C_j} G_i \delta n_i \, dC_j}{\int_{C_j} \delta n \, dC_j} \tag{40}$$

If delamination growth occurs only on partial boundary, that is, δn is greater than zero on certain boundary ΔC_j belonged to C_j and it equals to zero on residual boundary $C_j - \Delta C_j$, then the average energy release rate of this growth is

$$G_a = -\frac{\sum_{i=1}^{4} \int_{\Delta C_j} G_i \delta n_i \, dC_j}{\int_{\Delta C_j} \delta n \, dC_j} \tag{41}$$

Obviously, the average energy release rate relates to the mode of the delamination growth, but it is generally difficult to anticipate the actual mode of delamination growth because of that δn is a unknown continuous function. So it is difficult to calculate directly the average energy release rate by using Eq. (41). Now, supposing ΔC_j is a small segment including a given point and letting ΔC_j infinitely diminish to approach the point, then from Eq. (41) the energy release rate of any point can be given as

$$G = -\lim_{\Delta C_j \to 0} \frac{\sum_{i=1}^{4} \int_{\Delta C_j} G_i \delta n_i \, dC_j}{\int_{\Delta C_j} \delta n \, dC_j} \tag{42}$$

From Eq. (42), it can be seen that the G represents the distribution of energy release rate of any point on delaminated boundary.

For boundary C_1 , obviously having

$$\delta n_1 = -\delta n_2 = -\delta n_3 = -\delta n$$

and noticing Eq. (42), the energy release rate of any point on the boundary C_1 is

$$G_{C_1} = \lim_{\Delta C_1 \to 0} \frac{\int_{\Delta C_1} (G_1 - G_2 - G_3) \delta n \, dC_1}{\int_{\Delta C_1} \delta n \, dC_1} = G_1 - G_2 - G_3 \tag{43}$$

For boundary C_2, similarly having

$$\delta n_4 = -\delta n_2 = -\delta n_3 = -\delta n$$

and also noticing Eq. (42), the energy release rate of any point on the boundary C_2 is

$$G_{C_2} = \lim_{\Delta C_2 \to 0} \frac{\int_{\Delta C_2} (G_4 - G_2 - G_3) \delta n \, dC_2}{\int_{\Delta C_2} \delta n \, dC_2} = G_4 - G_2 - G_3 \tag{44}$$

Once the energy release rate being calculated, we can judge whether the delamination growth occurs according to the critical value G^c of energy release rate, that is, there is no delamination growth when the energy release rate is less than G^c and delamination growth will occur when the energy release rate is greater than G^c. In order to calculate the energy release rate, the displacements u_i, v_i, w_i must be first obtained through applying the central finite-difference method to Eq. (32) and their corresponding conditions (33)–(36). Then, the energy release rate of the delaminated cylindrical shell can be determined by adopting Eqs. (29) and (42). In the following numerical examples, the displacement solutions are not presented because the emphasis is given to the growth characteristics of delamination, rather than to the deformation pattern of delaminated cylindrical shell under external pressure.

4. Numerical results and discussion

For the sake of simplification, in present study, only the delamination growth of axisymmetrical laminated cylindrical shells is calculated.

Before discussing delamination growth caused by external radial load, the transverse deformation of region Ω_2 is first calculated for isotropic film delaminated cylindrical shell in order to validate the present analytical method. The delaminated cylindrical shell is subjected to radial uniform load only in region Ω_2. In this case, the region Ω_2 can be treated as a cylindrical shell with its both ends clamped and the radial displacement of this shell (region Ω_2) can be obtained on basis of the literature [20]. The geometrical parameters of the delaminated region Ω_2 are $L/R = 5/3$, $R/h = 30$ and the Poisson ratio v is 0.3. When the radial load $Q(Q = qL^4/(A_{22}h^3))$ takes value 1.875×10^4, the variable curve of radial displacement w along coordinate x for region Ω_2 is presented in Fig. 2. From Fig. 2, a good agreement between these two sets of values is observed. Thereby it is validated that the present analytical method and calculating procedure are reliable.

In the following, the effects of delamination lengths and depths, the geometrical parameters, the material properties and the laminate stacking sequences on the energy release rate G are investigated. In all figures, the delamination is assumed to be located symmetrically with respect to both ends of the shell (i.e. $l = \frac{(1-\beta)L}{2}$), the vertical ordinate $\bar{G}$ is the ratio of energy release rate G and $E_{22}h^5/((1\text{-}v_{12}v_{21})L^4)$.

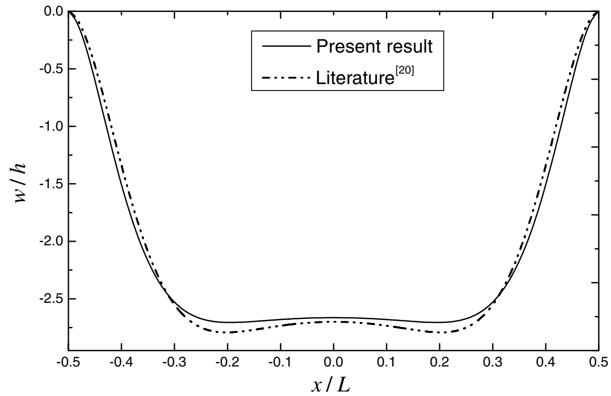


Fig. 2. Radial displacement of delaminated region Ω_2.

When the external radial load Q is taken as 1200, 1800 and 2500, respectively, the variable curves of energy release rate $\bar{G}$ with delamination length β are shown in Fig. 3. In this case, the cylindrical shell is isotropic ($v = 0.3$) and clamped on both ends. The geometrical parameters are $L/R = 5/3$, $R/h = 30$ and the delamination depth parameter $\alpha_2(= h_2/h)$ is 0.1. From Fig. 3, it can be seen that the energy release rate $\bar{G}$ increases with delamination length at first, but when β increases to a certain value, the energy release rate $\bar{G}$ reaches a maximum, and then it begins to decrease. This shows that the delamination growth is likely to be stable for delaminated cylindrical shell.

For different values of the delamination depth α_2, the variable curves of energy release rate $\bar{G}$ with external radial load Q are shown in Fig. 4. In this case, the cylindrical shell is isotropic ($v = 0.3$) and clamped on both ends. The geometrical parameters are $L/R = 5/3$, $R/h = 30$ and the delamination length parameter $\beta(\beta_2)$ is 0.2. From the figure, it can be seen that the energy release rate increases with delamination depth. This shows that the growth is easier to occur for deeper delamination.

For different ratio of the L/R, the variable curves of energy release rate $\bar{G}$ with external radial load Q are shown in Fig. 5. In this case, the cylindrical shell is isotropic ($v = 0.3$) and clamped on both ends. The geometrical parameters are $R/h = 30$ and the delamination parameters are $\alpha_2 = 0.2$, $\beta = 0.2$. From the figure, it can be seen that the energy release rate increases with the decrease of ratio of L/R. Thus, it can be concluded that the delamination growth is more difficult to occur when the ratio of L/R increases.

For delaminated cylindrical shell with different laid material, the variable curves of energy release rate with external radial load are shown in Fig. 6. In this numerical example, three kinds of materials are used. The first kind is

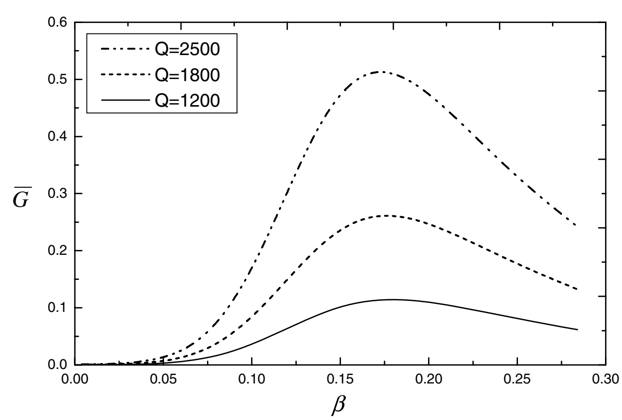


Fig. 3. Effect of delamination length on energy release rate.

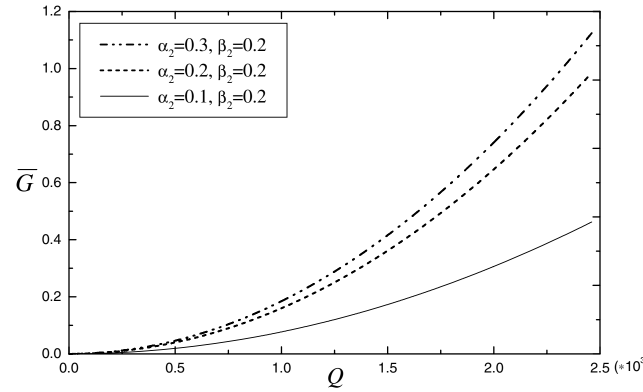


Fig. 4. Effect of delamination depth on energy release rate.

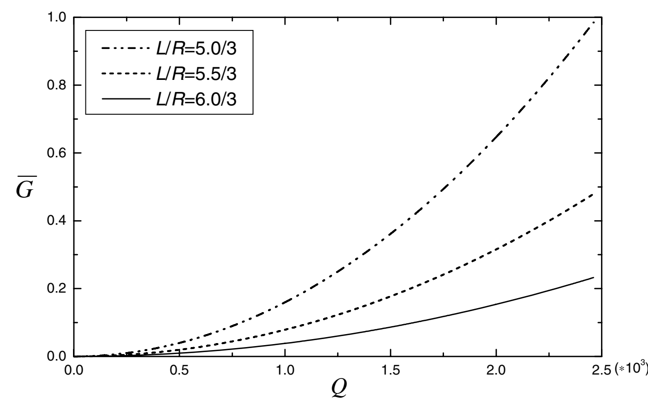


Fig. 5. Effect of the ratio of L/R on energy release rate.

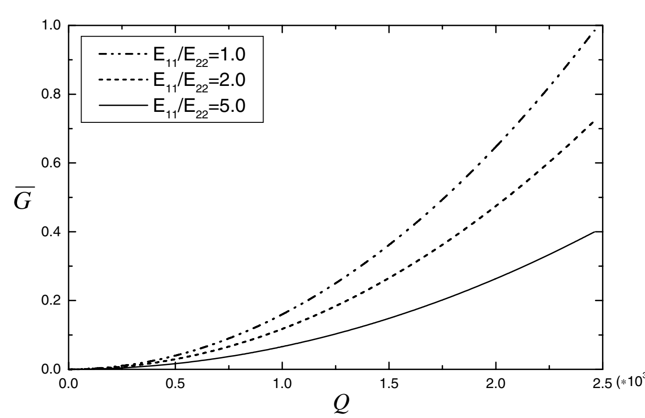


Fig. 6. Effect of E_{11}/E_{22} on energy release rate.

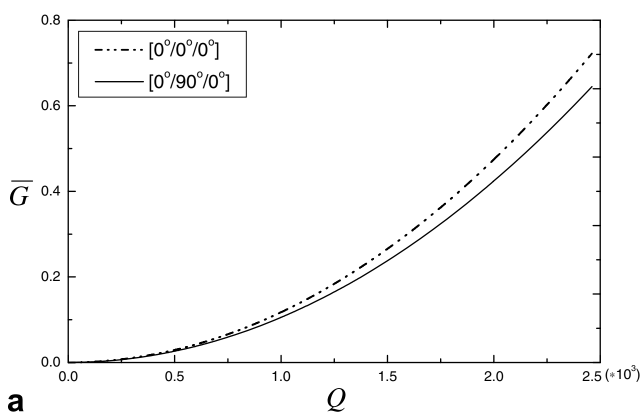
Table 1
Values of elastic constants

	E_{11}/E_{22}	G_{12}/E_{22}	V
Material 1	1.0	0.38	0.30
Material 2	2.0	0.43	0.30
Material 3	5.0	0.50	0.25

isotropic and the other two kinds are anisotropic. The used elastic constants of these materials are listed in Table 1. The geometrical parameters are $L/R = 5/3$, $R/h = 30$. The delamination parameters are $\alpha_2 = 0.2$, $\beta = 0.2$ and the stacking sequences of laminates are $[0°/0°/0°]_{10}$. From the figure, it can be seen that the energy release rate of material 1 is maximum and that of material 3 is minimum. This shows that the energy release rate of delaminated shell increases with the decrease of ratio of E_{11}/E_{22}. Thus, it can be concluded that the delamination growth is more difficult to occur when the ratio of E_{11}/E_{22} increases.

For different stacking sequences, the variable curves of energy release rate with external radial load are shown in Fig. 7. The geometrical parameters and the delamination parameters are the same with the above example. The used composites are material 2 and 3. From Fig. 7, it can be seen that the energy release rate of delaminated cylindrical shell with stacking sequences $[0°/90°/0°]_{10}$ is less than that of $[0°/0°/0°]_{10}$. This shows that the delamination growth is more difficult to occur when the anisotropism of material increases. Furthermore, comparing Fig. 7a and b, it can be seen that the effect of stacking sequences on energy release rate increases with the increase of E_{11}/E_{22}.

In addition, from the above all figures, it can be seen that the energy release rate of delaminated shell increases with external radial load. And this shows the delamination growth is easier to occur when the external radial load increases.



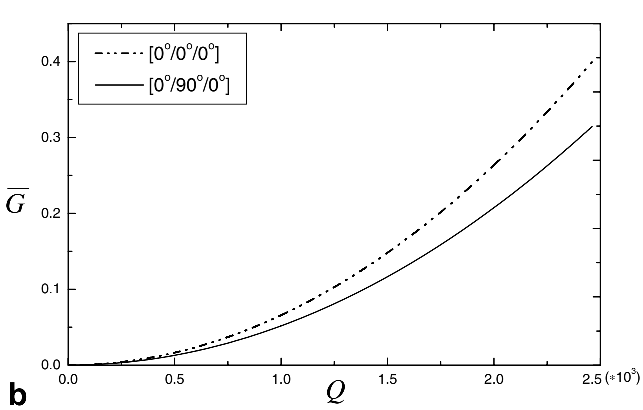


Fig. 7. Effect of stackling sequence on energy release rate. (a) Material 2, (b) Material 3.

5. Conclusions

Based on the variational principle of moving boundary and the Griffith criterion, the formula of energy release rate G along the delamination front are founded. In numerical examples, the effects of delamination sizes and depths, the geometrical parameters, the material properties and the laminate stacking sequences on delamination growth are discussed for axisymmetrical cylindrical shells. The following conclusions are drawn.

For a given external load Q, the energy release rate of the delaminated cylindrical shell increases with delamination length β at first, but when β increases to a particular value, the energy release rate reaches a maximum, and then it begins to decrease. The energy release rate and the possibility of delamination growth increase with the increase of delamination depth and external radial load, and decrease with the increase of the ratio of E_{11}/E_{22}, L/R and the anisotropism of material.

Acknowledgement

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Appendix A

The derivations of Eq. (25) are given as follows:

Using differential and integral calculus, Eq. (24) can be deduced as

$$\begin{aligned}\delta\Pi = \sum_{i=1}^{4}\int\int_{A_i}\Bigg\{&\left[\frac{\partial}{\partial x}(N_{xi}\delta u_i) - \frac{\partial}{\partial x}(N_{xi})\delta u_i\right] \\ &+ \left[\frac{\partial}{\partial x}(N_{xi}w_{i,x}\delta w_i) - \frac{\partial}{\partial x}(N_{xi}w_{i,x})\delta w_i\right] \\ &+ \left[\frac{\partial}{\partial y}(N_{yi}\delta v_i) - \frac{\partial}{\partial y}(N_{yi})\delta v_i\right] - \frac{N_{yi}}{R_i}\delta w_i \\ &+ \left[\frac{\partial}{\partial y}(N_{yi}w_{i,y}\delta w_i) - \frac{\partial}{\partial y}(N_{yi}w_{i,y})\delta w_i\right] \\ &+ \left[\frac{\partial}{\partial y}(N_{xyi}\delta u_i) - \frac{\partial}{\partial y}(N_{xyi})\delta u_i\right] \\ &+ \left[\frac{\partial}{\partial x}(N_{xyi}\delta v_i) - \frac{\partial}{\partial x}(N_{xyi})\delta v_i\right]\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{\partial}{\partial_x}(N_{xyi}w_{i,y}\delta w_i)-\frac{\partial}{\partial_x}(N_{xyi}w_{i,y})\delta w_i\right]\\
&+\left[\frac{\partial}{\partial_y}(N_{xyi}w_{i,x}\delta w_i)-\frac{\partial}{\partial_y}(N_{xyi}w_{i,x})\delta w_i\right]\\
&-\left[\frac{\partial}{\partial x}(M_{xi}\delta w_{i,x})-\frac{\partial}{\partial x}(M_{xi,x}\delta w_i)+\frac{\partial}{\partial x}(M_{xi,x})\delta w_i)\right]\\
&-\left[\frac{\partial}{\partial y}(M_{yi}\delta w_{i,y})-\frac{\partial}{\partial y}(M_{yi,y}\delta w_i)+\frac{\partial}{\partial y}(M_{yi,y})\delta w_i\right]\\
&-\left[\frac{\partial}{\partial x}(M_{xyi}\delta w_{i,y})-\frac{\partial}{\partial y}(M_{xyi,x}\delta w_i)+\frac{\partial}{\partial y}(M_{xyi,x})\delta w_i)\right]\\
&-\left[\frac{\partial}{\partial y}(M_{xyi}\delta w_{i,x})-\frac{\partial}{\partial x}(M_{xyi,y}\delta w_i)+\frac{\partial}{\partial x}(M_{xyi,y})\delta w_i)\right]\Bigg\}\mathrm{d}x\,\mathrm{d}y\\
&-\sum_{i=1}^{4}\int\int_{A_i}qH(x)\delta w_i\,\mathrm{d}x\,\mathrm{d}y\\
&-\sum_{i=1}^{4}\int\int_{A_i}T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}\delta w_i\,\mathrm{d}x\,\mathrm{d}y\\
&+\frac{1}{2}\sum_{i=1}^{4}\oint_{C_j}\Bigg\{N_{xi}\left(u_{i,x}+\frac{1}{2}w_{i,x}^2\right)\\
&+N_{yi}\left(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2\right)+N_{xyi}(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})\\
&+M_{xi}(-w_{i,xx})+M_{yi}(-w_{i,yy})+M_{xyi}(-2w_{i,xy})\\
&-qH(x)w_i-T_{\mathrm{c}}^{i}(q^*/2\pi R_{\mathrm{d}})w_i\Bigg\}\delta n_i\,\mathrm{d}C_j
\end{aligned}
\tag{A.1}
$$

Rearranging Eq. (A.1), it can be written as

$$
\begin{aligned}
\delta\Pi=&\sum_{i=1}^{4}\int\int_{A_i}\Bigg\{\left[-\frac{\partial}{\partial x}(N_{xi})-\frac{\partial}{\partial y}(N_{xyi})\right]\delta u_i\\
&+\left[-\frac{\partial}{\partial x}(N_{xyi})-\frac{\partial}{\partial y}(N_{yi})\right]\delta v_i\\
&+\Bigg[-\frac{\partial}{\partial x}(M_{xi,x})-\frac{\partial}{\partial y}(M_{yi,y})-\frac{\partial}{\partial y}(M_{xyi,x})\\
&-\frac{\partial}{\partial x}(M_{xyi,y})-\frac{\partial}{\partial x}(N_{xi}w_{i,x})-\frac{\partial}{\partial y}(N_{yi}w_{i,y})-\frac{\partial}{\partial x}(N_{xyi}w_{i,y})\\
&-\frac{\partial}{\partial y}(N_{xyi}w_{i,x})-\frac{N_{yi}}{R_i}-qH(x)-T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}\Bigg]\delta w_i\\
&+\Bigg[\frac{\partial}{\partial x}(N_{xi}\delta u_i)+\frac{\partial}{\partial x}(N_{xi}w_{i,x}\delta w_i)+\frac{\partial}{\partial x}(N_{xyi}\delta v_i)\\
&+\frac{\partial}{\partial x}(N_{xyi}w_{i,y}\delta w_i)-\frac{\partial}{\partial x}(M_{xi}\delta w_{i,x})+\frac{\partial}{\partial x}(M_{xi,x}\delta w_i)\\
&-\frac{\partial}{\partial x}(M_{xyi}\delta w_{i,y})+\frac{\partial}{\partial x}(M_{xyi,y}\delta w_i)\Bigg]\\
&+\Bigg[\frac{\partial}{\partial y}(N_{yi}\delta v_i)+\frac{\partial}{\partial y}(N_{yi}w_{i,y}\delta w_i)+\frac{\partial}{\partial y}(N_{xyi}\delta u_i)\\
&+\frac{\partial}{\partial y}(N_{xyi}w_{i,x}\delta w_i)-\frac{\partial}{\partial y}(M_{yi}\delta w_{i,y})+\frac{\partial}{\partial y}(M_{yi,y}\delta w_i)\\
&+\frac{\partial}{\partial y}(M_{xyi,x}\delta w_i)-\frac{\partial}{\partial y}(M_{xyi}\delta w_{i,x})\Bigg]\Bigg\}\mathrm{d}x\mathrm{d}y\\
&+\frac{1}{2}\sum_{i=1}^{4}\oint_{C_j}\Bigg\{N_{xi}\left(u_{i,x}+\frac{1}{2}w_{i,x}^2\right)+N_{yi}\left(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2\right)\\
&+N_{xyi}(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})+M_{xi}(-w_{i,xx})\\
&+M_{yi}(-w_{i,yy})+M_{xyi}(-2w_{i,xy})-qH(x)w_i\\
&-T_{\mathrm{c}}^{i}(q^*/2\pi R_{\mathrm{d}})w_i\Bigg\}\delta n_i\,\mathrm{d}C_j
\end{aligned}
\tag{A.2}
$$

The above equation can be further written as

$$
\begin{aligned}
\delta\Pi=&\sum_{i=1}^{4}\int\int_{A_i}\Bigg\{\left[-\frac{\partial}{\partial x}(N_{xi})-\frac{\partial}{\partial y}(N_{xyi})\right]\delta u_i\\
&+\left[-\frac{\partial}{\partial x}(N_{xyi})-\frac{\partial}{\partial y}(N_{yi})\right]\delta v_i\\
&+\Bigg[-\frac{\partial}{\partial x}(M_{xi,x})-\frac{\partial}{\partial y}(M_{yi,y})-\frac{\partial}{\partial y}(M_{xyi,x})-\frac{\partial}{\partial x}(M_{xyi,y})\\
&-\frac{\partial}{\partial x}(N_{xi}w_{i,x})-\frac{\partial}{\partial y}(N_{yi}w_{i,y})-\frac{\partial}{\partial x}(N_{xyi}w_{i,y})\\
&-\frac{\partial}{\partial y}(N_{xyi}w_{i,x})-\frac{N_{yi}}{R_i}-qH(x)-T_{\mathrm{c}}^{i}\frac{q^*}{2\pi R_{\mathrm{d}}}\Bigg]\delta w_i\Bigg\}\mathrm{d}x\,\mathrm{d}y\\
&+\sum_{i=1}^{4}\oint_{C_j}\{N_{xi}\delta u_i+N_{xi}w_{i,x}\delta w_i+N_{xyi}\delta v_i+N_{xyi}w_{i,y}\delta w_i\\
&-M_{xi}\delta w_{i,x}+M_{xi,x}\delta w_i-M_{xyi}\delta w_{i,y}+M_{xyi,y}\delta w_i\}\big|_{C_j}\mathrm{d}y\\
&+\{N_{yi}\delta v_i+N_{yi}w_{i,y}^{\delta}w_i+N_{xyi}\delta u_i+N_{xyi}w_{i,x}^{\delta}w_i-M_{yi}\delta w_{i,y}\\
&+M_{yi,y}\delta w_i+M_{xyi,x}\delta w_i-M_{xyi}\delta w_{i,x}\}\big|_{C_j}\mathrm{d}x\\
&+\frac{1}{2}\sum_{i=1}^{4}\oint_{C_j}\Bigg\{N_{xi}\left(u_{i,x}+\frac{1}{2}w_{i,x}^2\right)+N_{yi}\left(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2\right)\\
&+N_{xyi}(u_{i,y}+v_{i,x}+w_{i,x}w_{i,y})+M_{xi}(-w_{i,xx})+M_{yi}(-w_{i,yy})\\
&+M_{xyi}(-2w_{i,xy})-qH(x)w_i-T_{c}^{i}(q^*/2\pi R_{d})w_i\Bigg\}\delta n_i dC_j
\end{aligned}
\tag{A.3}
$$

Assuming ϑ_i is the angle between the normal direction n_i of a certain point on boundary C_j and $\theta=0$ (that is the axial direction x), then

$$
\mathrm{d}x=-\sin(\vartheta_i-\theta)\cdot dC_j,\quad \mathrm{d}y=\cos(\vartheta_i-\theta)\cdot dC_j \tag{A.4}
$$

For the delaminated cylindrical shell, the normal direction n_i of delamination growth is consistent with the axial direction x, that is $\vartheta_i-\theta=0$. Therefore, Eq. (A.3) can be written as

$$
\begin{aligned}
\delta\Pi=&\sum_{i=1}^{4}\int\int_{A_i}\Bigg\{\left[-\frac{\partial}{\partial x}(N_{xi})-\frac{\partial}{\partial y}(N_{xyi})\right]\delta u_i\\
&+\left[-\frac{\partial}{\partial x}(N_{xyi})-\frac{\partial}{\partial y}(N_{yi})\right]\delta v_i\\
&+\Bigg[-\frac{\partial}{\partial x}(M_{xi,x})-\frac{\partial}{\partial y}(M_{yi,y})-\frac{\partial}{\partial y}(M_{xyi,x})\\
&-\frac{\partial}{\partial x}(M_{xyi,y})-\frac{\partial}{\partial x}(N_{xi}w_{i,x})-\frac{\partial}{\partial y}(N_{yi}w_{i,y})\\
&-\frac{\partial}{\partial x}(N_{xyi}w_{i,y})-\frac{\partial}{\partial y}(N_{xyi}w_{i,x})-\frac{N_{yi}}{R_i}\\
&-qH(x)-T_{c}^{i}\frac{q^*}{2\pi R_{d}}\Bigg]\delta w_i\Bigg\}\mathrm{d}x\,\mathrm{d}y\\
&+\sum_{i=1}^{4}\oint_{C_j}\{N_{xi}\delta u_i+N_{xi}w_{i,x}\delta w_i+N_{xyi}\delta v_i\\
&+N_{xyi}w_{i,y}\delta w_i-M_{xi}\delta w_{i,x}+M_{xi,x}\delta w_i\\
&-M_{xyi}\delta w_{i,y}+M_{xyi,y}\delta w_i\}\big|_{C_j}dC_j\\
&+\frac{1}{2}\sum_{i=1}^{4}\oint_{C_j}\Bigg\{N_{xi}\left(u_{i,x}+\frac{1}{2}w_{i,x}^2\right)+N_{yi}\left(v_{i,y}-\frac{w_i}{R_i}+\frac{1}{2}w_{i,y}^2\right)
\end{aligned}
$$

$$+ N_{xyi}(u_{i,y} + v_{i,x} + w_{i,x}w_{i,y}) + M_{xi}(-w_{i,xx}) + M_{yi}(-w_{i,yy}) + M_{xyi}(-2w_{i,xy}) - qH(x)w_i - T_c^i(q^*/2\pi R_d)w_i\Big\}\delta n_i dC_j \tag{A.5}$$

As the variation of u_i, v_i, w_i is carried out on variable boundary C_j, we have

$$\begin{aligned}
\delta u_i|_{C_j} &= \delta(u_i|_{C_j}) - \frac{\partial u_i}{\partial n}\Big|_{C_j} \cdot \delta n_i,\\
\delta v_i|_{C_j} &= \delta(v_i|_{C_j}) - \frac{\partial v_i}{\partial n}\Big|_{C_j} \cdot \delta n_i\\
\delta w_i|_{C_j} &= \delta(w_i|C_j) - \frac{\partial w_i}{\partial n}\Big|_{C_j} \cdot \delta n_i,\\
\delta w_{i,x}|_{C_j} &= \delta(w_{i,x}|_{C_j}) - \frac{\delta w_{i,x}}{\delta n}|_{C_j} \cdot \delta n_i\\
\delta w_{i,y}|_{C_j} &= \delta(w_{i,y}|_{C_j}) - \frac{\partial w_{i,y}}{\partial n}\Big|_{C_j} \cdot \delta n_i
\end{aligned} \tag{A.6}$$

Using Eq. (A.6), Eq. (A.5) can be written as

$$\begin{aligned}
\delta\Pi = {} & \sum_{i=1}^{4} \int\int_{A_i} \Big\{ \Big[-\frac{\partial}{\partial x}(N_{xi}) - \frac{\partial}{\partial y}(N_{xyi}) \Big] \delta u_i \\
& + \Big[-\frac{\partial}{\partial x}(N_{xyi}) - \frac{\partial}{\partial y}(N_{yi}) \Big] \delta v_i \\
& + \Big[-\frac{\partial}{\partial x}(M_{xi,x}) - \frac{\partial}{\partial y}(M_{yi,y}) - \frac{\partial}{\partial y}(M_{xyi,x}) - \frac{\partial}{\partial x}(M_{xyi,y}) \\
& - \frac{\partial}{\partial x}(N_{xi}w_{i,x}) - \frac{\partial}{\partial y}(N_{yi}w_{i,y}) - \frac{\partial}{\partial x}(N_{xyi}w_{i,y}) \\
& - \frac{\partial}{\partial y}(N_{xyi}w_{i,x}) - \frac{N_{yi}}{R_i} - qH(x) - T_c^i \frac{q^*}{2\pi R_d} \Big] \delta w_i \Big\} \mathrm{d}x\,\mathrm{d}y \\
& + \sum_{i=1}^{4} \oint_{C_j} \Big\{ N_{xi}\delta(u_i|_{C_j}) + N_{xyi}\delta(v_i|_{C_j}) \\
& + (M_{xi,x} + M_{xyi,y} + N_{xi}w_{i,x} + N_{xyi}w_{i,y})\delta(w_i|_{C_j}) \\
& - M_{xi}\delta(w_{i,x}|_{C_j}) - M_{xyi}\delta(w_{i,y}|_{C_j}) \Big\} dC_j \\
& + \sum_{i=1}^{4} \oint_{C_j} \Big\{ \Big[N_{xi}\Big(-\frac{\partial u_i}{\partial n}\Big|_{C_j} \Big) + N_{xyi}\Big(-\frac{\partial v_i}{\partial n}\Big|_{C_j} \Big) \\
& + (M_{xi,x} + M_{xyi,y} + N_{xi}w_{i,x} + N_{xyi}w_{i,y}) \cdot \Big(-\frac{\partial w_i}{\partial n}\Big|_{C_j} \Big) \\
& + M_{xi}\frac{\partial w_{i,x}}{\partial n}\Big|_{C_j} + M_{xyi}\frac{\partial w_{i,y}}{\partial n}\Big|_{C_j} \Big] \cdot \delta n_i dC_j \\
& + \frac{1}{2}\sum_{i=1}^{4} \oint_{C_j} \Big\{ N_{xi}(u_{i,x} + \frac{1}{2}w_{i,x}^2) + N_{yi}(v_{i,y} - \frac{w_i}{R_i} + \frac{1}{2}w_{i,y}^2) \\
& + N_{xyi}(u_{i,y} + v_{i,x} + w_{i,x}w_{i,y}) + M_{xi}(-w_{i,xx}) \\
& + M_{yi}(-w_{i,yy}) + M_{xyi}(-2w_{i,xy}) - qH(x)w_i \\
& - T_c^i(q^*/2\pi R_d)w_i \Big\} \delta n_i dC_j
\end{aligned} \tag{A.7}$$

Eq. (A.7) is written as

$$\delta\Pi = \delta\Pi_1 + \delta\Pi_2 \tag{A.8}$$

where $\delta\Pi_1$ and $\delta\Pi_2$ are listed in Eqs. (26) and (27).

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