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Nonlinear dynamic response of piezoelastic laminated plates considering damage effects

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Abstract

Based on the Talreja's damage model with tensor valued internal state variables and the classical geometric nonlinear theory, the constitutive relations of the composite uni-ply plate with damage are derived. Then the nonlinear dynamic equations of the piezoelastic laminated plates considering damage effects are obtained. By using the finite difference method and the Newmark scheme, these equations are solved. Numerical results show the effects of damage and electric loads on the nonlinear dynamic response of the laminated piezoelastic plates.

Keywords: Piezoelastic laminated plate; Nonlinear dynamic response; Damage effect; Finite difference method

1. Introduction

The research of piezoelectric laminated structures has received considerable attention in recent years. Some investigations on the vibration of piezoelectric laminated plates have been done. Wang and Rogers [1] presented a model based on classical plate theory for laminated plates with spatially distributed piezoelectric patches. Tzou and Gadre [2] analyzed a multi-layered thin shell coupled with piezoelectric shell actuators for distributed vibration controls. Xu et al. [3] analyzed the free vibration of piezothermoelectric laminated plates based on the 3D theory. Mitchell and Reddy [4] proposed the theory of the piezoelectric laminated plates by using classical plate theory and simple third-order theory, respectively. Artel and Becker [5] analyzed the effect of electromechanical coupling on the interlaminar stresses and the electric field strengths at free edges of laminated plates with piezoelectric material properties. Benjeddou et al. [6] studied the free vibration of simply-supported piezoelectric adaptive plates by using an exact sandwich formulation. Rao and Sunar [7] developed a finite element formulation of piezothermoelastic media and integrated it with the distributed sensing and control of intelligent structures. Tauchert [8] applied Nowacki's general theory to piezothermoelastic laminated plates and obtained the static solutions. Tzou and Bao [9] presented a geometrical nonlinear theory of a piezothermoelastic laminated shell under the actions of mechanical, electric and thermal fields. Kapuria and Achary [10] applied a new coupled consistent third-order theory to hybrid piezoelectric plates and assessed the new and existing third-order theories. Mannini and Gaudenzi [11] developed multi-layer higher-order finite elements for the analysis of the stress concentration at the free edge of the active elements of a piezoelectric composite. However, various forms of damage will emerge easily in the piezoelastic laminated structures. And the emergence and evolution of the damage will reduce the stiffness of structures and lead to the change of static/dynamic behaviors. But, there is only little research on the static/dynamic behaviors of the piezoelastic laminated plates including damage effects.

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In the present study, on the basis of the Talreja's damage model with tensor valued internal state variables, the constitutive relations of the composite uni-ply plate with damage are derived. Then adopting the classical geometric nonlinear theory, the nonlinear dynamic equations of piezoelastic laminated plates with damage are obtained. The unknown functions are separated by using the finite difference method in the space domain, and the time is equally divided into small time segments. Then all the nonlinear items are linearized, and the separated equations are iterated to seek solutions. Finally, the effects of damage and electric loads on the nonlinear dynamic responses of the piezoelastic laminated plates are extensively discussed.

2. Basic equations

2.1. Constitutive relations

Consider a N layers piezoelastic laminated plate having thickness h, length a in the x-direction, width b in the y-direction shown in Fig. 1. The reference surface defined by $z=0$ is set on the middle surface of the undeformed plate. The plate is subjected to dynamic transverse load $q(x,y,t)$ combined with electrical load. Denote $\bar{z}_k$ and h_k as the z-coordinate of the mid-surface and the thickness of the kth layer, respectively.

In the present study, only damage effects in composite layers are taken into account. Talreja adopted a second-order tensor to describe the damage. Supposing the lth crack in the characteristic volume is characterized by a symmetric second-order tensor $\omega^{(l)}$, and damage variables can be defined as [12]

$$\omega_{ij}=\sum_{l=1}^{L}\omega_{ij}^{(l)}=\frac{1}{V}\sum_{l=1}^{L}\int_{s^{(l)}}d^{(l)}n_i^{(l)}n_j^{(l)}\,\mathrm{d}s \tag{1}$$

in which, $n^{(l)}$ is the unit vector that is vertical to the crack surface, $s^{(l)}$ is the crack surface, V is the capacity of the characteristic volume, L is the number of cracks in the characteristic volume, and $d^{(l)}$ is a scalar function which can be written as

$$d^{(l)}=f(A^{(l)},a^{(l)}) \tag{2}$$

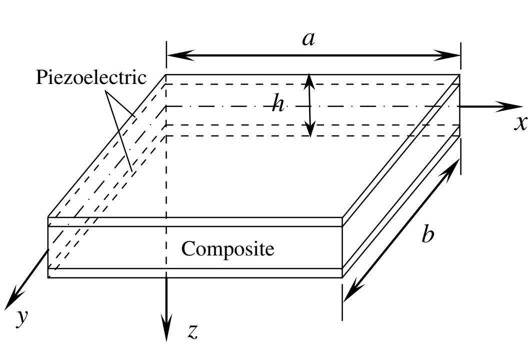

Fig. 1. Configuration of piezoelastic laminated plates.

where $A^{(l)}$ and $a^{(l)}$ are the area of the crack surface and the characteristic dimension of the crack, respectively.

Helmholtz free energy can be written as the invariant function of elastic strains and damage variables. When the damage of the fiber-reinforced composite material with orthotropic property is induced by cracks in the matrix, the bases of invariant can be presented as [13]

$$\begin{aligned}&\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},\varepsilon_{23}^2,\varepsilon_{31}^2,\varepsilon_{12}^2,\varepsilon_{12}\varepsilon_{23}\varepsilon_{31},\omega_{11}^m,\omega_{22}^m,\omega_{33}^m,(\omega_{23}^m)^2,(\omega_{31}^m)^2,\\&(\omega_{12}^m)^2,\omega_{12}^m\omega_{23}^m\omega_{31}^m,\varepsilon_{23}\omega_{23}^m,\varepsilon_{31}\omega_{31}^m,\varepsilon_{12}\omega_{12}^m,\omega_{23}^m\varepsilon_{12}\varepsilon_{13},\omega_{31}^m\varepsilon_{32}\varepsilon_{12},\\&\omega_{12}^m\varepsilon_{13}\varepsilon_{23},\varepsilon_{23}\omega_{12}^m\omega_{13}^m,\varepsilon_{31}\omega_{32}^m\omega_{12}^m,\varepsilon_{12}\omega_{13}^m\omega_{23}^m\end{aligned} \tag{3}$$

in which, $m=1,2,\ldots n$, n is the number of the cracks' direction in the material.

Consider a fiber-reinforced composite uni-ply plate. The local coordinate system o-123 is adopted, in which axis 1 is parallel to fibrous direction, axis 2 is vertical to fibrous direction, and axis 3 is vertical to the mid-surface. According to the fundamental assumption on the plate $\varepsilon_{13}=\varepsilon_{23}=0$ and applying Voigt notation to describe strains and damage variables, the bases of invariant can be further written as

$$\begin{aligned}&\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_6^2,\omega_1^m,\omega_2^m,\omega_3^m,(\omega_4^m)^2,(\omega_5^m)^2,(\omega_6^m)^2,\omega_4^m\omega_5^m\omega_6^m,\varepsilon_6\omega_6^m,\\&\varepsilon_6\omega_4^m\omega_5^m\quad(m=1,2,\ldots,n)\end{aligned} \tag{4}$$

Helmholtz free energy can be expressed as a quadratic expression of strains and a linear expression of damage variables for small strain problem, i.e.

$$\begin{aligned}\rho\psi={}&C_1^0\varepsilon_1^2+C_2^0\varepsilon_1\varepsilon_2+C_3^0\varepsilon_2^2+C_4^0\varepsilon_6^2+C_5^0\varepsilon_3^2+C_6^0\varepsilon_1\varepsilon_3+C_7^0\varepsilon_2\varepsilon_3\\&+\sum_{m=1}^{n}[C_1^m\varepsilon_1^2\omega_1^m+C_2^m\varepsilon_1^2\omega_2^m+C_3^m\varepsilon_1^2\omega_3^m+C_4^m\varepsilon_2^2\omega_1^m\\&+C_5^m\varepsilon_2^2\omega_2^m+C_6^m\varepsilon_2^2\omega_3^m+C_{15}^m\varepsilon_3^2\omega_1^m+C_{16}^m\varepsilon_3^2\omega_2^m+C_{17}^m\varepsilon_3^2\omega_3^m\\&+C_7^m\varepsilon_6^2\omega_1^m+C_8^m\varepsilon_6^2\omega_2^m+C_9^m\varepsilon_6^2\omega_3^m+C_{10}^m\varepsilon_1\varepsilon_2\omega_1^m\\&+C_{11}^m\varepsilon_1\varepsilon_2\omega_2^m+C_{12}^m\varepsilon_1\varepsilon_2\omega_3^m+C_{18}^m\varepsilon_1\varepsilon_3\omega_1^m+C_{19}^m\varepsilon_1\varepsilon_3\omega_2^m\\&+C_{20}^m\varepsilon_1\varepsilon_3\omega_3^m+C_{21}^m\varepsilon_2\varepsilon_3\omega_1^m+C_{22}^m\varepsilon_2\varepsilon_3\omega_2^m+C_{23}^m\varepsilon_2\varepsilon_3\omega_3^m\\&+C_{13}^m\varepsilon_1\varepsilon_6\omega_6^m+C_{14}^m\varepsilon_2\varepsilon_6\omega_6^m+C_{24}^m\varepsilon_3\varepsilon_6\omega_6^m]+P_0\\&+P_1(\varepsilon_p,\omega_q^m)+P_2(\omega_q^m)\end{aligned} \tag{5}$$

in which, C_i^0 $(i=1,2\ldots,7)$ are the material constants without damage, C_i^m $(i=1,2\ldots,24,\ m=1,2\ldots,n)$ are the material constants with damage, ρ is mass density, P_0 is a constant, P_1 is a linear function of strains, and P_2 is a linear function of damage variables. Assuming the directions of the cracks in the uni-ply plate are identical, i.e. $n=1$, then stresses can be written as

$$\sigma_p=\frac{\partial(\rho\psi)}{\partial\varepsilon_p}=[C_{pq}^{\,0}+C_{pq}]\varepsilon_q \tag{6}$$

where $[C_{pq}^0]$ and $[C_{pq}]$ are symmetric matrices having the forms as follows

$$[C_{pq}^0] = \begin{bmatrix} 2C_1^0 & C_2^0 & C_6^0 & 0 \\ & 2C_3^0 & C_7^0 & 0 \\ & & 2C_5^0 & 0 \\ & & & 2C_4^0 \end{bmatrix}$$

$$[C_{pq}] = \begin{bmatrix} 2C_1\omega_1 + 2C_2\omega_2 + 2C_3\omega_3 & C_{10}\omega_1 + C_{11}\omega_2 + C_{12}\omega_3 & C_{18}\omega_1 + C_{19}\omega_2 + C_{20}\omega_3 & C_{13}\omega_6 \\ & 2C_4\omega_1 + 2C_5\omega_2 + 2C_6\omega_3 & C_{21}\omega_1 + C_{22}\omega_2 + C_{23}\omega_3 & C_{14}\omega_6 \\ & & 2C_{15}\omega_1 + 2C_{16}\omega_2 + 2C_{17}\omega_3 & C_{24}\omega_6 \\ & & & 2C_7\omega_1 + 2C_8\omega_2 + 2C_9\omega_3 \end{bmatrix}$$

In the above expressions, superscript $m = 1$ is omitted.

For the plate with matrix cracks that is vertical to the fibrous direction, in all damage variables only ω_1 is not zero, then coefficient matrix in (6) is simplified as

$$[C_{pq}^0 + C_{pq}] = \begin{bmatrix} 2C_1^0 + 2C_1\omega_1 & C_2^0 + C_{10}\omega_1 & C_6^0 + C_{18}\omega_1 & 0 \\ & 2C_3^0 + 2C_4\omega_1 & C_7^0 + C_{21}\omega_1 & 0 \\ & & 2C_5^0 + 2C_{15}\omega_1 & 0 \\ & & & 2C_4^0 + 2C_7\omega_1 \end{bmatrix} \tag{7}$$

Because all cracks are parallel to the coordinate planes 2–3, its effect on stiffness in this coordinate plane can be neglected. Then the last matrix can be further simplified as

$$[C_{pq}^0 + C_{pq}] = \begin{bmatrix} 2C_1^0 + 2C_1\omega_1 & C_2^0 + C_{10}\omega_1 & C_6^0 + C_{18}\omega_1 & 0 \\ & 2C_3^0 & C_7^0 + C_{21}\omega_1 & 0 \\ & & 2C_5^0 & 0 \\ & & & 2C_4^0 + 2C_7\omega_1 \end{bmatrix} \tag{8}$$

Letting $\sigma_3 = 0$, the constitutive relation of the composite uni-ply plate with damage in the status of plane stress can be obtained as follows

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [C_{pq}^0 + C_{pq}] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \tag{9}$$

where

$$[C_{pq}^0] = \begin{bmatrix} 2C_1^0 - \frac{(C_6^0)^2}{2C_5^0} & C_2^0 - \frac{C_6^0 C_7^0}{2C_5^0} & 0 \\ & 2C_3^0 - \frac{(C_7^0)^2}{2C_5^0} & 0 \\ & & 2C_4^0 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} c_{11} & c_{12} & 0 \\ & c_{22} & 0 \\ & & c_{66} \end{bmatrix}$$

$$[C_{pq}] = \begin{bmatrix} 2C_1 - \frac{C_6^0 C_{18}}{C_5^0} & C_{10} - \frac{C_6^0 C_{21} + C_7^0 C_{18}}{2C_5^0} & 0 \\ & -\frac{C_7^0 C_{21}}{C_5^0} & 0 \\ & & 2C_7 \end{bmatrix} \omega_1 \stackrel{\text{def}}{=} \begin{bmatrix} d_{11} & d_{12} & 0 \\ & d_{22} & 0 \\ & & d_{66} \end{bmatrix} \omega_1 \tag{10}$$

The constitutive relation is available to the bending problem of laminated plates. The stress–strain relations in the kth layer of laminated plate considering damage effect are expressed as follows

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_k = \left(\begin{bmatrix} c_{11}^k & c_{12}^k & 0 \\ & c_{22}^k & 0 \\ & & c_{66}^k \end{bmatrix} + \begin{bmatrix} d_{11}^k & d_{12}^k & 0 \\ & d_{22}^k & 0 \\ & & d_{66}^k \end{bmatrix} \omega_1^k \right) \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}_k \stackrel{\text{def}}{=} \begin{bmatrix} E_{11}^k & E_{12}^k & 0 \\ & E_{22}^k & 0 \\ & & E_{66}^k \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}_k \tag{11}$$

In the present research, Kachanvo damage evolution law [14] is adopted for an arbitrary point in the kth layer with damage:

$$\frac{\partial \omega_1^k}{\partial t} = \begin{cases} B^k \left(\frac{\sigma_{eq}^k}{1-\omega_1^k} \right)^{n^k} & \sigma_{eq}^k \geqslant \sigma_d^k \\ 0 & \sigma_{eq}^k < \sigma_d^k \end{cases} \tag{12}$$

where B^k and n^k are the material constants, σ_{eq}^k is an equivalent stress which is based on certain failure criterion, σ_d^k is the stress threshold value at which the damage begins to grow.

The constitutive relationship of the piezoelectric layers in the status of plane stress can be written as follows

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_k = \begin{bmatrix} Q_{11}^k & Q_{12}^k & 0 \\ & Q_{22}^k & 0 \\ & & Q_{66}^k \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}_k - \begin{bmatrix} 0 & 0 & e_{31}^k \\ 0 & 0 & e_{32}^k \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}_k \tag{13}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{31}^k & e_{32}^k & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}_k - \begin{bmatrix} \in_{11}^k & 0 & 0 \\ 0 & \in_{22}^k & 0 \\ 0 & 0 & \in_{33}^k \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}_k \tag{14}$$

where D_i and E_i represent the electric displacement and electric field, respectively. Q_{ij}^k, e_{ij}^k and $\in_{ii}^k$ denote the kth piezoelectric layer stiffness constants, piezoelectric stress constants and dielectric constants, respectively.

The relations between the electric fields E_x,E_y,E_z and the electric potential ϕ in the cartesian coordinate system are defined by

$$E_x = -\phi_{,x}, \quad E_y = -\phi_{,y}, \quad E_z = -\phi_{,z} \tag{15}$$

For the piezoelastic laminated plate, only thickness direction electric field E_z is dominant. If the voltage applied to the kth layer with piezoelectric effect in the thickness only, then

$$E_z^k = V_k/h_k \tag{16}$$

where V_k is the applied voltage across the kth layer and h_k is the thickness of the layer.

2.2. Nonlinear strain–displacement relations and nonlinear dynamic equations

Letting u, v and w as the displacement components of an arbitrary point on the mid-surface along the direction of x, y and z, respectively. According to classical nonlinear theory, the strain components ε_x^0, ε_y^0 and γ_{xy}^0 of the mid-surface can be written as

$$\varepsilon_x^0 = u_{,x} + \frac{1}{2}w_{,x}^2, \quad \varepsilon_y^0 = v_{,y} + \frac{1}{2}w_{,y}^2, \quad \gamma_{xy}^0 = u_{,y} + v_{,x} + w_{,x}w_{,y} \tag{17}$$

and the curvatures κ_x, κ_y and κ_{xy} of mid-surface as

$$\kappa_x = -w_{,xx}, \quad \kappa_y = -w_{,yy}, \quad \kappa_{xy} = -2w_{,xy} \tag{18}$$

then the nonlinear strain–displacement relations are expressed as follows

$$\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \quad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \quad \gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy} \tag{19}$$

Suppose piezoelastic laminated plates are symmetric with regard to the mid-surface and the fibers of every layer are parallel to the coordinate axis x or y. Moreover, assume the damage variable remains constant through the thickness of damaged layer [12]. Denote N_x, N_y, N_{xy} as the membrane stress resultants and M_x, M_y, M_{xy} as the stress couples of the plate. According to the classical nonlinear plate theory, the nonlinear governing equations of the piezoelastic laminated plates can be written as

$$\begin{aligned} &N_{x,x} + N_{xy,y} = 0 \\ &N_{xy,x} + N_{y,y} = 0 \\ &M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} + q = I_\rho w_{,tt} \end{aligned} \tag{20}$$

where $I_\rho = \sum_{k=1}^{N} \rho_k h_k$, and according to the classical laminated plate theory, the following constitutive equations can be obtained

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \mathrm{d}z = \begin{bmatrix} A_{11} & A_{12} & 0 \\ & A_{22} & 0 \\ & & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - \begin{Bmatrix} N_x^p \\ N_y^p \\ N_{xy}^p \end{Bmatrix} \tag{21}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} z \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \mathrm{d}z = \begin{bmatrix} D_{11} & D_{12} & 0 \\ & D_{22} & 0 \\ & & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} M_x^p \\ M_y^p \\ M_{xy}^p \end{Bmatrix} \tag{22}$$

where superscript p represents the component induced by the electric field. The stiffness coefficients A_{ij}, $D_{ij}(i,j = 1, 2, 6)$ of the piezoelastic laminated plate are defined as follows

$$A_{ij} = \sum_{k=1}^{N} F_{ij}^k (z_k - z_{k-1}),$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} F_{ij}^k (z_k^3 - z_{k-1}^3) \quad (F_{ij}^k = E_{ij}^k \text{ or } Q_{ij}^k, \quad i,j = 1,2,6) \tag{23}$$

where z_k and z_{k-1} denote the z coordinate values of the upper and nether surface for the kth layer, respectively. The resultants and couples due to the piezoelectric effect can be written as

$$N_x^p = \sum_{k=1}^{N} e_{31}^k V_k, \; N_y^p = \sum_{k=1}^{N} e_{32}^k V_k, \quad N_{xy}^p = 0$$

$$M_x^p = \frac{1}{2}\sum_{k=1}^{N} e_{31}^k V_k (z_k + z_{k-1}),$$

$$M_y^p = \frac{1}{2}\sum_{k=1}^{N} e_{32}^k V_k (z_k + z_{k-1}), \quad M_{xy}^p = 0 \tag{24}$$

Introduce the following dimensionless parameters:

$$\begin{aligned} &\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad U = \frac{u}{a}, \quad V = \frac{v}{b}, \quad W = \frac{w}{h}, \quad \overline{A}_{11} = \frac{A_{11}}{c_{22}^1 h}, \\ &\overline{A}_{12} = \frac{A_{12}}{c_{22}^1 h}, \quad \overline{A}_{22} = \frac{A_{22}}{c_{22}^1 h}, \quad \overline{A}_{66} = \frac{A_{66}}{c_{22}^1 h}, \quad \overline{D}_{11} = \frac{D_{11}}{c_{22}^1 h^3}, \\ &\overline{D}_{12} = \frac{D_{12}}{c_{22}^1 h^3}, \quad \overline{D}_{22} = \frac{D_{22}}{c_{22}^1 h^3}, \quad \overline{D}_{66} = \frac{D_{66}}{c_{22}^1 h^3}, \quad \alpha_k = \frac{c_{11}^k}{c_{22}^k}, \\ &\beta_k = \frac{c_{12}^k}{c_{22}^k}, \quad \eta_k = \frac{c_{66}^k}{c_{22}^k}, \quad \xi_k = \frac{e_{32}^k}{e_{31}^k}, \quad \lambda_1 = \frac{h}{a}, \quad \lambda_2 = \frac{h}{b}, \\ &\bar{h}_k = \frac{h_k}{h}, \quad \varsigma_k = \frac{z_k}{h}, \quad \zeta_k = \frac{\bar{z}_k}{h}, \quad \bar{\sigma}_{eq}^k = \frac{\sigma_{eq}^k}{c_{22}^k}, \quad \bar{\sigma}_d^k = \frac{\sigma_d^k}{c_{22}^k}, \\ &\bar{d}_{11}^k = \frac{d_{11}^k}{c_{22}^k}, \quad \bar{d}_{12}^k = \frac{d_{12}^k}{c_{22}^k}, \quad \bar{d}_{22}^k = \frac{d_{22}^k}{c_{22}^k}, \quad \bar{d}_{66}^k = \frac{d_{66}^k}{c_{22}^k}, \\ &\tau = t\sqrt{\frac{c_{22}^1}{I_\rho h}}, \quad Q = \frac{q}{c_{22}^1}, \quad \overline{V}_k = \frac{e_{31}^k V_k}{c_{22}^1 h} \end{aligned} \tag{25}$$

in which, superscript k represents the material constant of the kth layer. By using Eqs. (9), (10), (15), (16) and (20)–(25), the dimensionless nonlinear governing equations of piezoelastic symmetric cross-ply laminated plates with damage are obtained and expressed in terms of U, V and W as follows

$$
\begin{aligned}
&\overline{A}_{11,\xi}(U_{,\xi}+\frac{1}{2}\lambda_1^2W_{,\xi}^2)+\overline{A}_{11}(U_{,\xi\xi}+\lambda_1^2W_{,\xi}W_{,\xi\xi})\\
&\quad+\overline{A}_{12,\xi}(V_{,\eta}+\frac{1}{2}\lambda_2^2W_{,\eta}^2)+\overline{A}_{12}(V_{,\xi\eta}+\lambda_2^2W_{,\eta}W_{,\xi\eta})\\
&\quad+\frac{\lambda_2}{\lambda_1}\overline{A}_{66,\eta}\left(\frac{\lambda_2}{\lambda_1}U_{,\eta}+\frac{\lambda_1}{\lambda_2}V_{,\xi}+\lambda_1\lambda_2W_{,\xi}W_{,\eta}\right)\\
&\quad+\overline{A}_{66}\left(\frac{\lambda_2^2}{\lambda_1^2}U_{,\eta\eta}+V_{,\xi\eta}+\lambda_2^2W_{,\eta}W_{,\xi\eta}+\lambda_2^2W_{,\xi}W_{,\eta\eta}\right)=0\\
&\overline{A}_{12,\eta}\left(U_{,\xi}+\frac{1}{2}\lambda_1^2W_{,\xi}^2\right)+\overline{A}_{12}(U_{,\xi\eta}+\lambda_1^2W_{,\xi}W_{,\xi\eta})\\
&\quad+\overline{A}_{22,\eta}\left(V_{,\eta}+\frac{1}{2}\lambda_2^2W_{,\eta}^2\right)+\overline{A}_{22}\left(V_{,\eta\eta}+\lambda_2^2W_{,\eta}W_{,\eta\eta}\right)\\
&\quad+\frac{\lambda_1}{\lambda_2}\overline{A}_{66,\xi}\left(\frac{\lambda_2}{\lambda_1}U_{,\eta}+\frac{\lambda_1}{\lambda_2}V_{,\xi}+\lambda_1\lambda_2W_{,\xi}W_{,\eta}\right)\\
&\quad+\overline{A}_{66}\left(U_{,\xi\eta}+\frac{\lambda_1^2}{\lambda_2^2}V_{,\xi\xi}+\lambda_1^2W_{,\eta}W_{\xi\xi}+\lambda_1^2W_{,\xi}W_{,\xi\eta}\right)=0\\
&-\lambda_1^4(\overline{D}_{11,\xi\xi}W_{,\xi\xi}+2\overline{D}_{11,\xi}W_{,\xi\xi\xi}+\overline{D}_{11}W_{,\xi\xi\xi\xi})\\
&\quad-\lambda_1^2\lambda_2^2(\overline{D}_{12,\xi\xi}W_{\eta\eta}+2\overline{D}_{12,\xi}W_{,\xi\eta\eta}+\overline{D}_{12}W_{,\xi\xi\eta\eta})\\
&\quad-4\lambda_1^2\lambda_2^2(\overline{D}_{66,\xi\eta}W_{,\xi\eta}+\overline{D}_{66,\xi}W_{,\xi\eta\eta}+\overline{D}_{66,\eta}W_{,\xi\xi\eta}+\overline{D}_{66}W_{,\xi\xi\eta\eta})\\
&\quad-\lambda_1^2\lambda_2^2(\overline{D}_{12,\eta\eta}W_{,\xi\xi}+2\overline{D}_{12,\eta}W_{\xi\xi\eta}+\overline{D}_{12}W_{,\xi\xi\eta\eta})\\
&\quad-\lambda_2^4(\overline{D}_{22,\eta\eta}W_{,\eta\eta}+2\overline{D}_{22,\eta}W_{,\eta\eta\eta}+\overline{D}_{22}W_{,\eta\eta\eta\eta})\\
&\quad+\lambda_1^2[\overline{A}_{11}\left(U_{,\xi}+\frac{1}{2}\lambda_1^2W_{,\xi}^2\right)+\overline{A}_{12}\left(V_{,\eta}+\frac{1}{2}\lambda_2^2W_{,\eta}^2\right)\\
&\quad-\sum_{k=1}^{N}\overline{V}_k]W_{,\xi\xi}+2\lambda_1\lambda_2\overline{A}_{66}\left(\frac{\lambda_2}{\lambda_1}U_{,\eta}+\frac{\lambda_1}{\lambda_2}V_{,\xi}+\lambda_1\lambda_2W_{,\xi}W_{,\eta}\right)\\
&\quad\times W_{,\xi\eta}+\lambda_2^2\overline{A}_{12}\left(U_{,\xi}+\frac{1}{2}\lambda_1^2W_{,\xi}^2\right)+\overline{A}_{22}\left(V_{,\eta}+\frac{1}{2}\lambda_2^2W_{,\eta}^2\right)\\
&\quad-\sum_{k=1}^{N}\xi_k\overline{V}_k]W_{,\eta\eta}+Q(\xi,\eta,\tau)=W_{,\tau\tau}
\end{aligned}\tag{26}
$$

Suppose all the boundary of the piezoelastic laminated plates are simply supported, the dimensionless boundary conditions can be expressed as

$$
\begin{aligned}
&\xi=0,1:\quad V=0,\ \overline{A}_{11}\left(U_{,\xi}+\frac{1}{2}\lambda_1^2W_{,\xi}^2\right)-\sum_{k=1}^{N}\overline{V}_k=0\\
&\qquad W=0,\ -\lambda_1^2\overline{D}_{11}W_{,\xi\xi}-\frac{1}{2}\sum_{k=1}^{N}\overline{V}_k(\varsigma_k+\varsigma_{k-1})=0\\
&\eta=0,1:\quad U=0,\ \overline{A}_{22}\left(V_{,\eta}+\frac{1}{2}\lambda_2^2W_{,\eta}^2\right)-\sum_{k=1}^{N}\xi_k\overline{V}_k=0\\
&\qquad W=0,\ -\lambda_2^2\overline{D}_{22}W_{,\eta\eta}-\frac{1}{2}\sum_{k=1}^{N}\xi_k\overline{V}_k(\varsigma_k+\varsigma_{k-1})=0
\end{aligned}\tag{27}
$$

The dimensionless evolution equation can be written as

$$
\frac{\partial\omega_1^k}{\partial\tau}=\begin{cases}\overline{B}^k\left(\dfrac{\bar{\sigma}_{eq}^k}{1-\omega_1^k}\right)^{n^k} & \bar{\sigma}_{eq}^k\geqslant\bar{\sigma}_d^k\\ 0 & \bar{\sigma}_{eq}^k<\bar{\sigma}_d^k\end{cases}\tag{28}
$$

Taking the mid-surface normal stress of the kth layer as the equivalent stress σ_{eq}^k that is parallel to the fibrous direction, and if the fibers in the kth layer are parallel to the axis x, it can be presented as

$$
\begin{aligned}
\bar{\sigma}_{eq}^k=&(\alpha_k+\bar{d}_{11}^k\omega_1^k)\left(U_{,\xi}+\frac{1}{2}\lambda_1^2W_{,\xi}^2-\lambda_1^2\zeta_kW_{,\xi\xi}\right)\\
&+(\beta_k+\bar{d}_{12}^k\omega_1^k)\left(V_{,\eta}+\frac{1}{2}\lambda_2^2W_{,\eta}^2-\lambda_2^2\zeta_kW_{,\eta\eta}\right)
\end{aligned}\tag{29}
$$

nevertheless, if the fibers in the kth layer are parallel to the axis y, it is presented as

$$
\begin{aligned}
\bar{\sigma}_{eq}^k=&(\beta_k+\bar{d}_{12}^k\omega_1^k)\left(U_{,\xi}+\frac{1}{2}\lambda_1^2W_{,\xi}^2-\lambda_1^2\zeta_kW_{,\xi\xi}\right)\\
&+\left(1+\bar{d}_{22}^k\omega_1^k\right)\left(V_{,\eta}+\frac{1}{2}\lambda_2^2W_{,\eta}^2-\lambda_2^2\zeta_kW_{,\eta\eta}\right)
\end{aligned}\tag{30}
$$

3. Solution methodology

Suppose the dimensionless transverse load is taken as

$$Q=f(\tau)\sin\pi\xi\sin\pi\eta,f(\tau)=f_0\sin\theta\tau\tag{31}$$

where f_0 and θ represent the dimensionless amplitude and frequency of the transverse load, respectively. Since the load and the structure are symmetric, only one quarter of the plate needs be considered. And the domain of the problem is selected as $0\leqslant\xi\leqslant1/2$, $0\leqslant\eta\leqslant1/2$, which is separated into $m\times m$ lattices.

To seek the approximate solutions of the governing Eq. (26) which satisfied the boundary conditions (27), the unknown functions U, V and W are separated both for space and for time. The finite difference method is used for space, and the partial derivatives with respect to the space coordinate variables are replaced by difference form. The time τ is equally divided into small time segments $\Delta\tau$, and the whole equations are iterated to seek solutions. At each step of the iteration, the nonlinear items in the equations and the boundary conditions are linearized. For example, at the step J, the nonlinear items may be transformed to

$$(x\cdot y)_J=(x)_J\cdot(y)_{J_p}\tag{32}$$

where $(y)_{J_p}$ is the average value of those obtained in the preceding two iterations. For the initial step of the iteration, it can be determined by using the quadratic extrapolation, i.e.

$$(y)_{J_p}=A(y)_{J-1}+B(y)_{J-2}+C(y)_{J-3}\tag{33}$$

and for the different step of the iteration, the coefficients A, B and C can be expressed as follows

$$
\begin{aligned}
&J=1:\quad A=1,\quad B=0,\quad C=0\\
&J=2:\quad A=2,\quad B=-1,\quad C=0\\
&J\geqslant3:\quad A=3,\quad B=-3,\quad C=1
\end{aligned}\tag{34}
$$

Moreover, using the Newmark scheme, the inertia in Eq. (26) can be expressed as follows

$$(W_{,\tau\tau})_J = \frac{4(W_J - W_{J-1})}{(\Delta\tau)^2} - \frac{4(W_{,\tau})_{J-1}}{\Delta\tau} - (W_{,\tau\tau})_{J-1}$$
$$(W_{,\tau})_J = (W_{,\tau})_{J-1} + \frac{1}{2}[(W_{,\tau\tau})_{J-1} + (W_{,\tau\tau})_J]\Delta\tau$$
$$W_J = W_{J-1} + (W_{,\tau})_{J-1}\Delta\tau + \frac{1}{4}[(W_{,\tau\tau})_{J-1} + (W_{,\tau\tau})_J](\Delta\tau)^2 \quad (35)$$

For every time step, the iteration lasts until the difference of the present value and the former is smaller than 0.1%, then continue the calculation of the next step.

4. Numerical examples and discussion

4.1. Comparison study

To ensure the accuracy and effectiveness of the present method, two test examples were solved for free vibrations of piezoelastic laminated plates without considering damage effect and nonlinear dynamic response of isotropic plates without considering piezo-effect and damage effect.

Example 1. The effect of thickness-span ratio (h/a) on the fundamental frequencies of laminated piezoelastic plates without damage is investigated. Consider a piezoelastic laminated plate consist of the symmetrical eight-layer cross-ply [90/0/90/0/0/90/0/90] laminates and two PZT-5A layers. The plates are square of side length $a = 1$, and all the layers have the same thickness, the material properties [5] are given as $E_1 = 181.0$ GPa, $E_2 = 10.3$ GPa, $G_{12} = G_{13} = 7.17$ GPa, $G_{23} = 3.87$ GPa, $v_{12} = 0.28$, $\rho = 1580.0$ kg/m ^{3} for the graphite-epoxy and $E_{11} = E_{22} = 61.0$ GPa, $G_{12} = 22.6$ GPa, $v_{12} = 0.35$, $\rho = 7750$ kg/m^3, $e_{31} = e_{32} = 7.209$ C/m^2 for the PZT-5A. The fundamental frequencies are calculated and compared in Table 1 with three-dimensional (3-D) solutions and first-order shear-deformation (FSDPT) solutions. Table 1 shows that the present results are satisfactory up to the layerwise FSDPT validity limit, but larger than 3-D solutions because the nonlinear items in nonlinear dynamic equations are linearized in the solution methodology.

Example 2. The present problem is simplified and used to nonlinear dynamic response analysis for isotropic rectangular plate without considering piezo-effect and damage effect. A simply-supported movable three-layer symmetric cross-ply laminated plate having the same thickness and material constants is considered. The geometric and material parameters are given as $\lambda_1 = 0.1$, $\lambda_2 = 0.1$, $\alpha_1 = 10$, $\beta_1 = 0.3$, $\eta_1 = 1$, $f_0 = 0.004$, $\theta = 0.01$. The response curve of the center deflection W_0 vs. time for isotropic plate is presented in Fig. 2, and compared with the results from FEM. The close agreements between the present results and those of the FEM solution demonstrate the present method is accurate and effective.

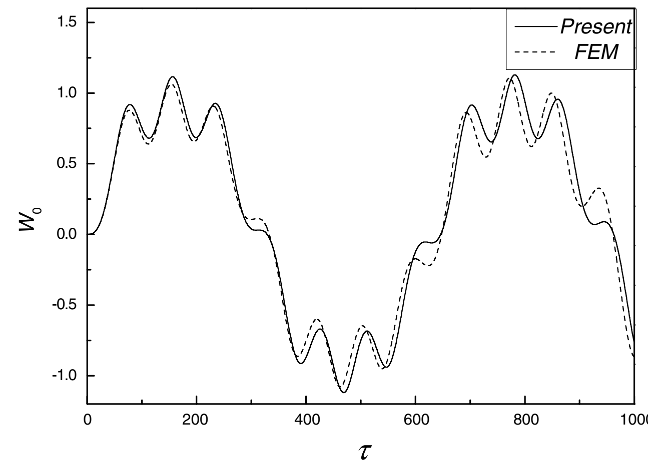

Fig. 2. Nonlinear dynamic response of center deflection vs. time for isotropic plate.

4.2. Parametric study

Consider a piezoelastic laminated plate consist of the symmetrical four-layer cross-ply [0/90/90/0] laminates including damage effects and two piezoelectric layers. Graphite/epoxy composite material and PVDF are selected for the substrate orthotropic layers and piezoelectric layers, and all layers have the identical thickness. The material properties adopted are$Q_{11} = 3.61$ GPa,$Q_{22} = 3.13$ GPa, $Q_{12} = 0.69$ GPa, $e_{31} = 32.075 \times 10^{-3}$ C/m^2, $e_{32} = -4.07 \times 10^{-3}$ C/m^2 and $\rho = 1800$ kg/m^3.

In all examples, the geometric parameters are given as $\lambda_1 = 0.1$, $\lambda_2 = 0.1$. When the damage effect is in consideration, the material parameters related to damage in all examples are taken as $\bar{B}^1 = 0.12$, $n^1 = 3$, $\bar{\sigma}_d^1 = 10^{-2}$, $\bar{d}_{11}^1 = -0.15$, $\bar{d}_{12}^1 = -0.15$, $\bar{d}_{22}^1 = -0.15$, $\bar{d}_{66}^1 = -0.15$.

Fig. 3 shows the effect of damage on the nonlinear dynamic response of the piezoelastic laminated plate under different electric loads, where W_0 is the central deflection of the plate, V_U and V_L represent the control voltages applied

Table 1
Comparison of fundamental frequencies of piezoelastic laminated square plate under different thickness-span ratios

$h/a = 0.01$			$h/a = 0.1$			$h/a = 0.2$		
Ref. [3] 3-D	Ref. [5] FSDPT	Present	Ref. [3] 3-D	Ref. [5] FSDPT	Present	Ref. [3] 3-D	Ref. [5] FSDPT	Present
268.86	283.93	293.2	2357.7	2516.7	2623.6	3648.0	3953.2	4053.6

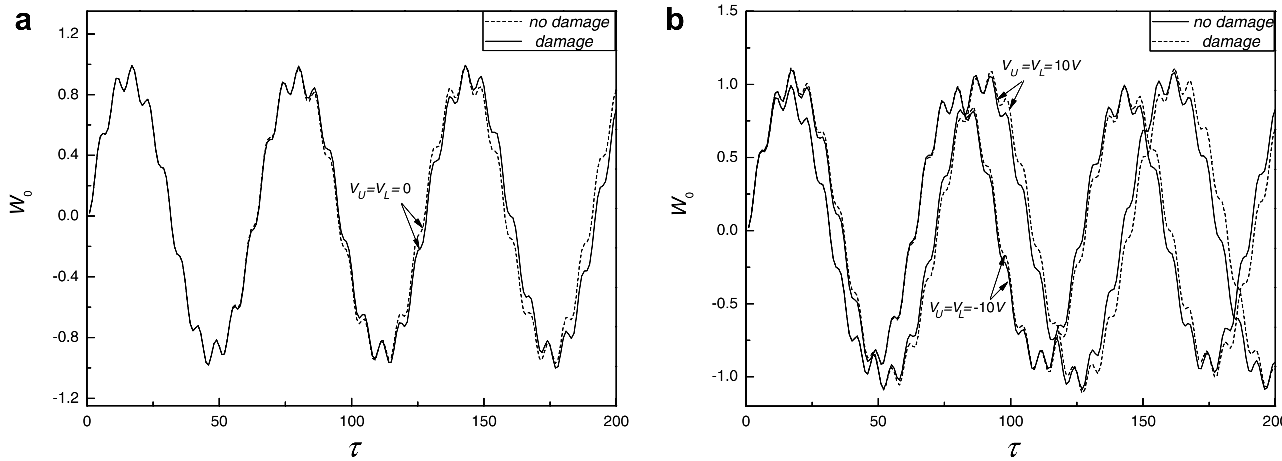

Fig. 3. Effect of damage on nonlinear dynamic response of piezoelastic laminated plate under different electric loads: (a) without electric load and (b) with electric load.

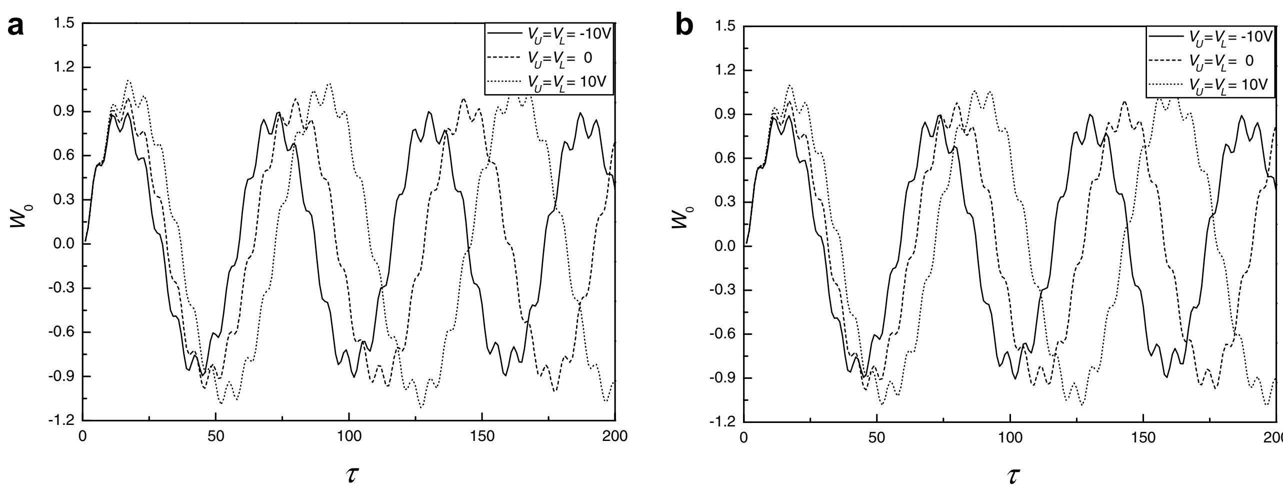

Fig. 4. Effect of electric load on nonlinear dynamic response of piezoelastic laminated plate: (a) without damage and (b) with damage.

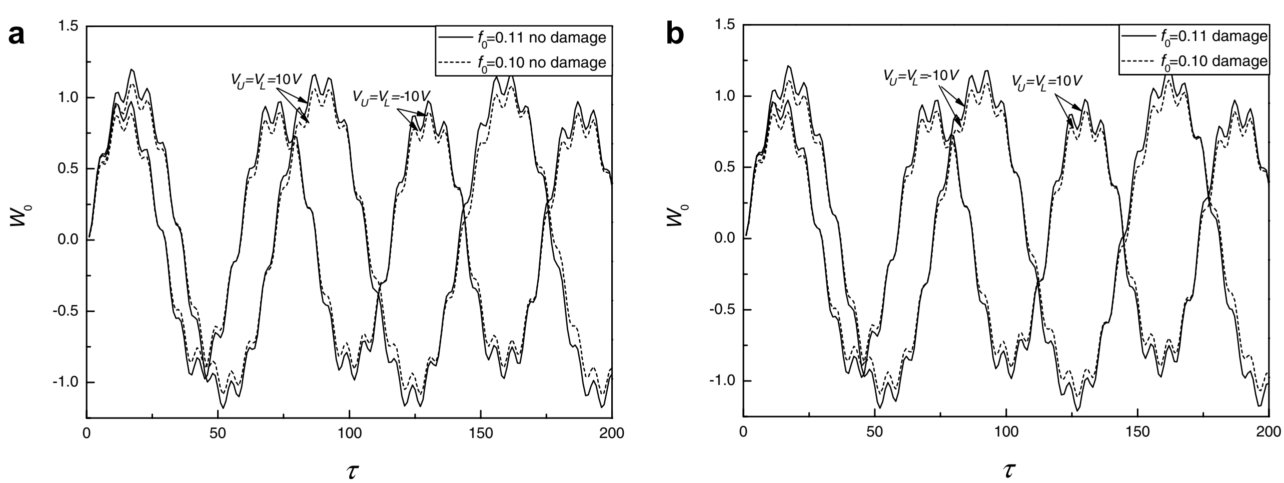

Fig. 5. Effect of amplitude of transverse load on nonlinear dynamic response of piezoelastic laminated plate under different electric loads: (a) without damage and (b) with damage.

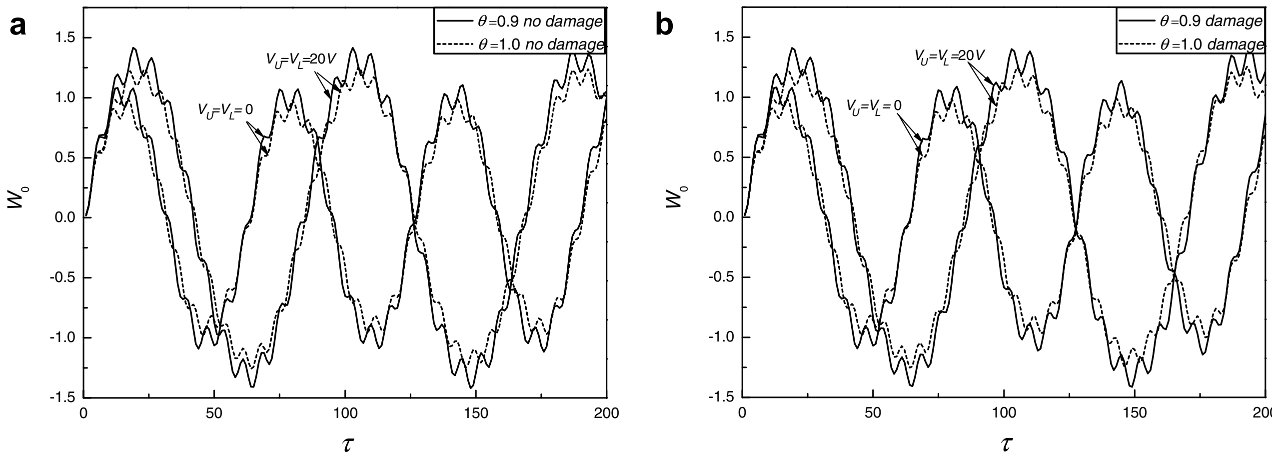

Fig. 6. Effect of frequency of transverse load on nonlinear dynamic response of piezoelastic laminated plate under different electric loads: (a) without damage and (b) with damage.

on the top and bottom piezoelectric layers, respectively. The dimensionless amplitude and frequency of the transverse load is taken as $f_0 = 0.1$, $\theta = 1.0$, respectively. It can be seen in Fig. 3 that the development of the damage becomes more apparent with the increase of time, the damage and the increment of control voltage values decrease the response frequencies and increase the response amplitude. So the conclusion can be drawn that the stiffness of the piezoelastic laminated plate decreases with the development of damage.

Fig. 4 shows the effect of electric load on the nonlinear dynamic response of the piezoelastic laminated plate with damage or without damage. The dimensionless amplitude and frequency of the transverse load is taken as $f_0 = 0.1$, $\theta = 1.0$, respectively. It can be shown that positive control voltage decreases the response frequencies and increases the response amplitude while the negative control voltage increases the response frequencies and decreases the response amplitude.

Fig. 5 shows the effect of the amplitude of the transverse load on the nonlinear dynamic response of the piezoelastic laminated plate with damage or without damage under different electric loads. It can be seen that, the larger the amplitude of the transverse load is, the faster the development of damage becomes. As indicated in Fig. 5, the increase of the amplitude of the transverse load increases the response amplitude. And this influence becomes apparent when positive control voltage acts upon the piezoelectric layers.

Fig. 6 shows the effect of the frequency of the transverse load on the nonlinear dynamic response of the piezoelastic laminated plate with damage or without damage under different electric loads. It can be shown that the response amplitudes of the piezoelastic laminated plate increase when the frequency of the transverse load decreases. And it also can be investigated that the larger positive control voltage is, the more apparent the variation of the response amplitude becomes under two different frequencies of the transverse loads.

5. Conclusion

Nonlinear dynamic response of piezoelastic laminated plates considering damage effect is investigated in this paper. The analytical solutions are presented by using the finite difference method and the Newmark scheme. The main conclusions can be drawn as follows. The damage and the increment of control voltage values decrease the response frequencies and increase the response amplitudes. The increment of the amplitude of the transverse load increases the response amplitudes of the laminated plate and this influence becomes apparent when positive control voltage acts upon the piezoelectric layers. The reduction of the frequency of the transverse load increases the response amplitudes of the piezoelastic laminated plate.

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