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RESEARCH PAPER

A continuum damage model for piezoelectric materials

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Abstract In this paper, a constitutive model is proposed for piezoelectric material solids containing distributed cracks. The model is formulated in a framework of continuum damage mechanics using second rank tensors as internal variables. The Helmhotlz free energy of piezoelectric materials with damage is then expressed as a polynomial including the transformed strains, the electric field vector and the tensorial damage variables by using the integrity bases restricted by the initial orthotropic symmetry of the material. By using the Talreja's tensor valued internal state damage variables as well as the Helmhotlz free energy of the piezoelectric material, the constitutive relations of piezoelectric materials with damage are derived. The model is applied to a special case of piezoelectric plate with transverse matrix cracks. With the Kirchhoff hypothesis of plate, the free vibration equations of the piezoelectric rectangular plate considering damage is established. By using Galerkin method, the equations are solved. Numerical results show the effect of the damage on the free vibration of the piezoelectric plate under the close-circuit condition, and the present results are compared with those of the three-dimensional theory.

Keywords Tensor valued internal state variables · Continuum damage mechanics · Damage constitutive relations · Piezoelectric plate

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1 Introduction

The piezoelectric materials have considerable applications in intelligent structures due to the intrinsic direct and converse piezoelectric effects, as sensors or actuators for the control of the active shape or vibration of structures. Defects such as micro cracks, voids, dislocations and delaminations are introduced in the piezoelectric materials during manufacturing and poling process. These defects greatly affect the electric, dielectric, elastic, mechanical and piezoelectric properties of the piezoelectric materials. When subjected to mechanical and electrical loads, these defects may grow in size and cracks may propagate, leading to premature mechanical or electrical failure of the materials. Therefore, it is important to understand the growth of these defects and the overall effect of these defects on the average mechanical and electrical properties of piezoelectric materials so that reliable service life predictions of the structure can be made. Such analysis is yet to be reliably accomplished and the progress depends on how well one can formulate constitutive relationships for anisotropic solids incorporating deformation and its interaction with distributed defects. The study should be guided by earlier work on elastic composites with distributed cracks and by many investigations on the vibration of piezoelectric plates.

Damages in fiber-reinforced composite materials were widely investigated, and many theories have been established and used to predict the life of composite structures. Moore and Dillard [1] observed the time dependent growth of transverse cracks in graphite/epoxy and Kevlar/epoxy cross-ply laminates at room temperature. Extensive studies on deformation, fracture, and damage of linear and nonlinear behavior of monolithic and composite materials were carried out by Schapery [2] using thermodynamics of irreversible processes. Luo and Daniel [3] found that the macroscopic mechanical behavior of unidirectional fiber-reinforced brittle matrix composites can be correlated explicitly with the microscopic deformation and damage. Deng and Nemat-Nasser [4] proposed an analytical and numerical method to determine the microscopic stress and strain distributions in structures with microcracks. Zheng and Fu [5] analyzed the nonlinear vibration of symmetric angle-ply laminated viscoelastic plates with damage.

Modeling and analysis of multilayer piezoelectric beams and plates have reached a relative maturity as attested by the numerous papers. Mindlin [6] presented a theory of piezoelectric crystal plates considering shear and bending. Tiersten [7] developed a general piezoelectric nonlinear theory, with vibration equations of different piezoelectric crystals. Chandrashekhara and Tenneti [8] and Zhou et al. [9] investigated the dynamic control of laminated piezoelectric plates by FE method. Wang and Rogers [10] presented a model based on classical plate theory for laminated plates with spatially distributed piezoelectric patches. Tzou and Gadre [11] analyzed thin laminates coupled with shell actuators for distributed vibration control. Xu et al. [12] analyzed the free vibration of laminated piezothermoelectric plate based on the 3D theory. Mitchell and Reddy [13] proposed a theory of the laminated piezoelectric plates by using classical plate theory and simple third-order theory separately. Rao and Sunar [14] developed a finite element formulation of piezothermoelastic media and integrated it with the distributed sensors and controllers of intelligent structures. Tauchert [15] applied Nowacki's general theory to piezothermoelastic laminated plates and obtained static solutions. Zheng and Fu [16] analyzed the nonlinear dynamic stability for piezoelastic laminated plates with damage through applying the energy principle and Lagrange equation. Up to now, the dynamic and static problems of piezoelectric structures considering the damage influence have not been well investigated yet.

In the present paper, a new constitutive model for piezoelectric materials is proposed using the Talreja's tensor valued internal state damage variables and the Helmhotlz free energy of piezoelectric materials. This model is applied to a special case of the piezoelectric plate with transverse matrix cracks. Then, the constitutive relations for piezoelectric plates with damage are derived. Damage evolution is not incorporated in the present formulation directly. With the Kirchhoff hypothesis, the free vibration equation for the piezoelectric rectangular plate with damage is established. In numerical calculations, the effects of the damage on the natural frequencies of the piezoelectric plates with damage are examined and compare with the results of the three-dimensional theory.

2 Constitutive equations for damaged piezoelectric materials

Consider a representative volume element of a piezoelectric solid with damage entities in the form of microcracks, as

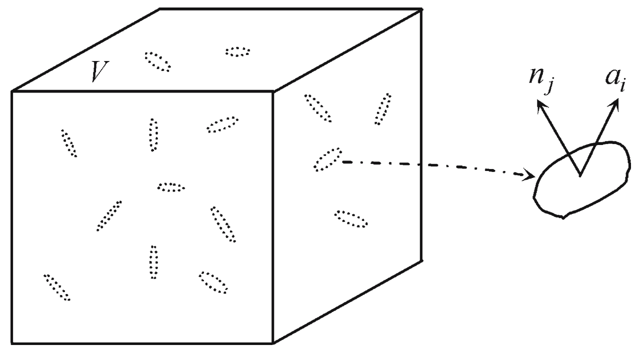

Fig. 1 Representative volume element with internal damage variables for piezoelectric materials

shown in Fig. 1. As discussed by Talreja, two vectors are used to define each damage entity. These are damage influence vector a_i and unit normal n_j to the damage entity surface. Damage influence vector represents an appropriately chosen effect of the damage entity on the surrounding medium. With these two vectors, a damage entity tensor is formed by taking an integral of the diad $a_i \cdot n_j$ over the surface of the damage entity.

$$d_{ij} = \int_S a_i \cdot n_j \mathrm{d}S, \quad i, j = 1, 2, 3. \tag{1}$$

Now if there are n distinct damage modes in the representative volume element (e.g., intralaminar cracks in different orientations etc.,) denoted by $k = 1, 2, \ldots, n$, a damage tensor can be defined for each mode as

$$\omega_{ij}^{k} = \frac{1}{V} \sum_{\vartheta_k} (d_{ij})_{\vartheta_k}, \tag{2}$$

where V is the volume of the representative volume element and ϑ_k represents the number of damage entities in the kth damage mode. The Tensor ω_{ij} is unsymmetric in general. However, we can decompose the vector a_i along the normal and tangential directions at any point on the surface of the damage entity and write

$$d_{ij} = d_{ij}^{1} + d_{ij}^{2}, \tag{3}$$

where $d_{ij}^{1} = \int_S a n_i \cdot n_j \mathrm{d}S$ and $d_{ij}^{2} = \int_S b m_i \cdot n_j \mathrm{d}S$, in which, a and b are the magnitudes of the normal and tangential projections of vector a_i, respectively, and vectors n_j and m_i are unit normal and tangential vectors, respectively. Thus the damage tensor ω_{ij} can be written as

$$\omega_{ij}^{k} = \omega_{ij}^{1k} + \omega_{ij}^{2k}, \tag{4}$$

where $\omega_{ij}^{1k} = \frac{1}{V} \sum_{\vartheta_k} (d_{ij}^{1})_{\vartheta_k}$ and $\omega_{ij}^{2k} = \frac{1}{V} \sum_{\vartheta_k} (d_{ij}^{2})_{\vartheta_k}$.

Physically, the first tensor represents the effects of crack opening on the surrounding medium whereas the second tensor represents the effects of sliding between two crack faces. In many situations, the sliding between crack faces can be neglected, e.g., for intralaminar cracks constrained by stiff plies, and hence we assume $\omega_{ij}^{2k} \equiv 0$. This implies $\omega_{ij}^{k} \equiv \omega_{ij}^{1k}$, which is a symmetric tensor.

For the case of damaged piezoelectric materials, where the damage is represented by internal state variables, the Helmholtz free energy can be written as a function of the transformed elastic strains, the electric field vector and damage internal variables.

$$H = H(\varepsilon_{ij}, E_i, \omega_{ij}^{k}). \tag{5}$$

The transformed stress components σ_{ij} and the electric displacement components D_i at any fixed damage state are now expressed as

$$\begin{aligned} \sigma_{ij} &= \frac{\partial H(\varepsilon_{ij}, E_i, \omega_{ij}^{k})}{\partial \varepsilon_{ij}}, \\ D_i &= -\frac{\partial H(\varepsilon_{ij}, E_i, \omega_{ij}^{k})}{\partial E_i}. \end{aligned} \tag{6}$$

When the damage induced by the cracks in the matrix of the piezoelectric material has the orthotropic property, the irreducible integrity bases for a scalar polynomial function of two symmetric second rank tensors can be expressed as [17]

$$\begin{aligned} &\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}^2, \varepsilon_{31}^2, \varepsilon_{12}^2, \varepsilon_{12}\varepsilon_{23}\varepsilon_{31}, \\ &\omega_{11}^k, \omega_{22}^k, \omega_{33}^k, (\omega_{23}^k)^2, (\omega_{31}^k)^2, (\omega_{12}^k)^2, \omega_{12}^k\omega_{23}^k\omega_{31}^k, \\ &\varepsilon_{23}\omega_{23}^k, \varepsilon_{31}\omega_{31}^k, \varepsilon_{12}\omega_{12}^k, \omega_{23}^k\varepsilon_{12}\varepsilon_{13}, \omega_{31}^k\varepsilon_{32}\varepsilon_{12}, \omega_{12}^k\varepsilon_{13}\varepsilon_{23}, \\ &\varepsilon_{23}\omega_{12}^k\omega_{13}^k, \varepsilon_{31}\omega_{32}^k\omega_{12}^k, \varepsilon_{12}\omega_{13}^k\omega_{23}^k, \\ &E_1, E_2, E_3, \\ &k = 1, 2, \ldots, n, \end{aligned} \tag{7}$$

where n is the number of the cracks' directions in the material.

For a piezoelectric single-layer plate, the local coordinate system o–123 is selected, in which 1, 2 denote the two principal directions of the piezoelectric plate, 3 is normal to the midsurface. According to the Kirchhoff hypothesis for plate $\varepsilon_{13} = \varepsilon_{23} = 0$ and with Voigt notations for strains and damage variables, the bases of invariant can be further written as

$$\begin{aligned} &\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_6^2, \omega_1^k, \omega_2^k, \omega_3^k, (\omega_4^k)^2, (\omega_5^k)^2, (\omega_6^k)^2, \\ &\omega_4^k\omega_5^k\omega_6^k, \varepsilon_6\omega_6^k, \varepsilon_6\omega_4^k\omega_5^k, \\ &E_1, E_2, E_3, \\ &k = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

Using the above stated irreducible integrity bases, Helmholtz free energy of piezoelectric materials can be expressed as a quadratic expression of the strains or the electric field intensity, a mixture quadratic expression of strains and electric field intensity and a linear expression of damage variables [18].

$$\begin{aligned} H = {} & C_1^0\varepsilon_1^2 + C_2^0\varepsilon_1\varepsilon_2 + C_3^0\varepsilon_2^2 + C_4^0\varepsilon_6^2 + C_5^0\varepsilon_1^2 \\ & + C_6^0\varepsilon_1\varepsilon_3 + C_7^0\varepsilon_2\varepsilon_3 - (\kappa_1^0E_1^2 + \kappa_2^0E_2^2 \\ & + \kappa_3^0E_3^2 + \kappa_4^0E_1E_2 + \kappa_5^0E_2E_3 + \kappa_6^0E_3E_1) \\ & - (e_1^0E_1\varepsilon_1 + e_2^0E_1\varepsilon_2 + e_3^0E_1\varepsilon_3 + e_4^0E_2\varepsilon_1 \\ & + e_5^0E_2\varepsilon_2 + e_6^0E_2\varepsilon_3 + e_7^0E_3\varepsilon_1 + e_8^0E_3\varepsilon_2 + e_9^0E_3\varepsilon_3) \\ & + \sum_{k=1}^{n}(C_1^k\varepsilon_1^2\omega_1^k + C_2^k\varepsilon_1^2\omega_2^k + C_3^k\varepsilon_1^2\omega_3^k \\ & + C_4^k\varepsilon_2^2\omega_1^k + C_5^k\varepsilon_2^2\omega_2^k + C_6^k\varepsilon_2^2\omega_3^k + C_7^k\varepsilon_3^2\omega_1^k \\ & + C_8^k\varepsilon_3^2\omega_2^k + C_9^k\varepsilon_3^2\omega_3^k + C_{10}^k\varepsilon_6^2\omega_1^k \\ & + C_{11}^k\varepsilon_6^2\omega_2^k + C_{12}^k\varepsilon_6^2\omega_3^k + C_{13}^k\varepsilon_1\varepsilon_2\omega_1^k + C_{14}^k\varepsilon_1\varepsilon_2\omega_2^k \\ & + C_{15}^k\varepsilon_1\varepsilon_2\omega_3^k + C_{16}^k\varepsilon_2\varepsilon_3\omega_1^k + C_{17}^k\varepsilon_2\varepsilon_3\omega_2^k \\ & + C_{18}^k\varepsilon_2\varepsilon_3\omega_3^k + C_{19}^k\varepsilon_3\varepsilon_1\omega_1^k + C_{20}^k\varepsilon_3\varepsilon_1\omega_2^k \\ & + C_{21}^k\varepsilon_3\varepsilon_1\omega_3^k + C_{22}^k\varepsilon_6\varepsilon_1\omega_6^k + C_{23}^k\varepsilon_6\varepsilon_2\omega_6^k \\ & + C_{24}^k\varepsilon_6\varepsilon_3\omega_6^k) - \sum_{k=1}^{n}(\kappa_1^kE_1^2\omega_1^k + \kappa_2^kE_1^2\omega_2^k \\ & + \kappa_3^kE_1^2\omega_3^k + \kappa_4^kE_2^2\omega_1^k + \kappa_5^kE_2^2\omega_2^k \\ & + \kappa_6^kE_3^2\omega_3^k + \kappa_7^kE_3^2\omega_1^k + \kappa_8^kE_3^2\omega_2^k \\ & + \kappa_9^kE_3^2\omega_3^k + \kappa_{10}^kE_1E_2\omega_1^k + \kappa_{11}^kE_1E_2\omega_2^k \\ & + \kappa_{12}^kE_1E_2\omega_3^k + \kappa_{13}^kE_2E_3\omega_1^k + \kappa_{14}^kE_2E_3\omega_2^k \\ & + \kappa_{15}^kE_2E_3\omega_3^k + \kappa_{16}^kE_3E_1\omega_1^k + \kappa_{17}^kE_3E_1\omega_2^k \\ & + \kappa_{18}^kE_3E_1\omega_3^k) - \sum_{k=1}^{n}(e_1^kE_1\varepsilon_1\omega_1^k + e_2^kE_1\varepsilon_1\omega_2^k \\ & + e_3^kE_1\varepsilon_1\omega_3^k + e_4^kE_1\varepsilon_2\omega_1^k + e_5^kE_1\varepsilon_2\omega_2^k \\ & + e_6^kE_1\varepsilon_2\omega_3^k + e_7^kE_1\varepsilon_3\omega_1^k + e_8^kE_1\varepsilon_3\omega_2^k \\ & + e_9^kE_1\varepsilon_3\omega_3^k + e_{10}^kE_2\varepsilon_1\omega_1^k + e_{11}^kE_2\varepsilon_1\omega_2^k \\ & + e_{12}^kE_2\varepsilon_1\omega_3^k + e_{13}^kE_2\varepsilon_2\omega_1^k + e_{14}^kE_2\varepsilon_2\omega_2^k \\ & + e_{15}^kE_2\varepsilon_2\omega_3^k + e_{16}^kE_2\varepsilon_3\omega_1^k + e_{17}^kE_2\varepsilon_3\omega_2^k \\ & + e_{18}^kE_2\varepsilon_3\omega_3^k + e_{19}^kE_3\varepsilon_1\omega_1^k + e_{20}^kE_3\varepsilon_1\omega_2^k \\ & + e_{21}^kE_3\varepsilon_1\omega_3^k + e_{22}^kE_3\varepsilon_2\omega_1^k + e_{23}^kE_3\varepsilon_2\omega_2^k \\ & + e_{24}^kE_3\varepsilon_2\omega_3^k + e_{25}^kE_3\varepsilon_3\omega_1^k + e_{26}^kE_3\varepsilon_3\omega_2^k \\ & + e_{27}^kE_3\varepsilon_3\omega_3^k + e_{28}^kE_1\varepsilon_6\omega_6^k + e_{29}^kE_2\varepsilon_6\omega_6^k \\ & + e_{30}^kE_3\varepsilon_6\omega_6^k) + P_0 + P_1(\varepsilon_p, \omega_q^k) \\ & + P_2(\omega_q^k) + P_3(E_i, \omega_q^k), \end{aligned} \tag{9}$$

where $C_i^0(i = 1, 2, \ldots, 7)$ are the material constants without damage, $\kappa_i^0(i = 1, 2, \ldots, 6)$ are the piezoelectric constants without damage, $e_i^0(i = 1, 2, \ldots, 9)$ are the permittivity matrix constants without damage, $C_i^k (i = 1, 2, \ldots, 24)$ are the material constants with damage; $\kappa_i^k (i = 1, 2, \ldots, 18)$ are the piezoelectric constants with damage, $e_i^k (i = 1, 2, \ldots, 30)$ are the permittivity matrix constants with damage, ρ is the density of the piezoelectric material, P_0 is a constant, P_1 is a linear function of strains, P_2 is a linear function of damage variables and P_3 is a linear function of the electric field intensity. Then the stresses and the electric displacements can be expressed as

$$\sigma_p = \frac{\partial H}{\partial \varepsilon_p} = \left[(C^0)_{pq} + \sum_{k=1}^{n} (C^k)_{pq} \right] \varepsilon_q - \left[(e^0)_{pm} + \sum_{k=1}^{n} (e^k)_{pm} \right] E_m, \quad p, q = 1, 2, 3, 6, \tag{10}$$

$$D_m = -\frac{\partial H}{\partial E_m} = \left[(e^0)_{qm} + \sum_{k=1}^{n} (e^k)_{qm} \right]^{\mathrm{T}} \varepsilon_q + \left[(\kappa^0)_{mn} + \sum_{k=1}^{n} (\kappa^k)_{mn} \right]^{\mathrm{T}} E_m, \quad m, n = 1, 2, 3,$$

where $(C^0)_{pq}$, $(C^k)_{pq}$, $(\kappa^0)_{mn}$ and $(\kappa^k)_{mn}$ are all symmetric matrixes in the forms of

$$(C^0)_{pq} = \begin{bmatrix} 2C_1^0 & C_2^0 & C_6^0 & 0 \\ & 2C_3^0 & C_7^0 & 0 \\ & & 2C_5^0 & 0 \\ & & & 2C_4^0 \end{bmatrix}, \quad p, q = 1, 2, 3, 6, \tag{11}$$

$$(C^k)_{pq} = \begin{bmatrix} 2C_1^k\omega_1^k + 2C_2^k\omega_2^k + 2C_3^k\omega_3^k & C_{13}^k\omega_2^k + C_{14}^k\omega_2^k + C_{15}^k\omega_3^k & C_{19}^k\omega_2^k + C_{20}^k\omega_2^k + C_{21}^k\omega_3^k & C_{22}^k\omega_6^k \\ & 2C_4^k\omega_1^k + 2C_5^k\omega_2^k + 2C_6^k\omega_3^k & C_{16}^k\omega_2^k + C_{17}^k\omega_2^k + C_{18}^k\omega_3^k & C_{23}^k\omega_6^k \\ & & 2C_7^k\omega_1^k + 2C_8^k\omega_2^k + 2C_9^k\omega_3^k & C_{24}^k\omega_6^k \\ & & & 2C_{10}^k\omega_1^k + 2C_{11}^k\omega_2^k + 2C_{12}^k\omega_3^k \end{bmatrix}, \tag{12}$$

$$(e^0)_{pm} = \begin{bmatrix} e_1^0 & e_4^0 & e_7^0 \\ e_2^0 & e_5^0 & e_8^0 \\ e_3^0 & e_6^0 & e_9^0 \\ 0 & 0 & 0 \end{bmatrix}, \quad m = 1, 2, 3, \tag{13}$$

$$(\kappa^0)_{mn} = \begin{bmatrix} 2\kappa_1^0 & \kappa_4^0 & \kappa_6^0 \\ & 2\kappa_2^0 & \kappa_5^0 \\ & & 2\kappa_3^0 \end{bmatrix}, \quad m, n = 1, 2, 3, \tag{14}$$

$$(e^k)_{pm} = \begin{bmatrix} e_1^k\omega_1^k + e_2^k\omega_2^k + e_3^k\omega_3^k & e_{10}^k\omega_1^k + e_{11}^k\omega_2^k + e_{12}^k\omega_3^k & e_{19}^k\omega_1^k + e_{20}^k\omega_2^k + e_{21}^k\omega_3^k \\ e_4^k\omega_1^k + e_5^k\omega_2^k + e_6^k\omega_3^k & e_{13}^k\omega_1^k + e_{14}^k\omega_2^k + e_{15}^k\omega_3^k & e_{22}^k\omega_1^k + e_{23}^k\omega_2^k + e_{24}^k\omega_3^k \\ e_7^k\omega_1^k + e_8^k\omega_2^k + e_9^k\omega_3^k & e_{16}^k\omega_1^k + e_{17}^k\omega_2^k + e_{18}^k\omega_3^k & e_{25}^k\omega_1^k + e_{26}^k\omega_2^k + e_{27}^k\omega_3^k \\ e_{28}^k\omega_6^k & e_{29}^k\omega_6^k & e_{30}^k\omega_6^k \end{bmatrix}, \tag{15}$$

$$(\kappa^k)_{mn} = \begin{bmatrix} 2\kappa_1^k\omega_1^k + 2\kappa_2^k\omega_2^k + 2\kappa_3^k\omega_3^k & \kappa_{10}^k\omega_1^k + \kappa_{11}^k\omega_2^k + \kappa_{12}^k\omega_3^k & \kappa_{16}^k\omega_1^k + \kappa_{17}^k\omega_2^k + \kappa_{18}^k\omega_3^k \\ & 2\kappa_4^k\omega_1^k + 2\kappa_5^k\omega_2^k + 2\kappa_6^k\omega_3^k & \kappa_{13}^k\omega_1^k + \kappa_{14}^k\omega_2^k + \kappa_{15}^k\omega_3^k \\ & & 2\kappa_7^k\omega_1^k + 2\kappa_8^k\omega_2^k + 2\kappa_9^k\omega_3^k \end{bmatrix}. \tag{16}$$

Assuming that there is only one damage mode in the representative volume element, the relations of the strains, the stresses, the electric field intensity and the electric displacements in Eq. (10) can be simplified as

$$\begin{aligned} \sigma_p &= [(C^0)_{pq} + (C)_{pq}]\varepsilon_q - [(e^0)_{pm} + (e)_{pm}]E_m, \\ D_m &= [(e^0)_{qm} + (e)_{qm}]^{\mathrm{T}}\varepsilon_q + [(\kappa^0)_{mn} + (\kappa)_{mn}]^{\mathrm{T}}E_m, \end{aligned} \tag{17}$$

where $(C^0)_{pq}$, $(e^0)_{pm}$ and $(\kappa^0)_{mn}$ are the same as before. The superscript $k = 1$ is omitted in $(C^k)_{pq}$, $(e^k)_{pm}$, $(\kappa^k)_{mn}$.

In the present study, the matrix cracks in the piezoelectric plate are parallel to the coordinate plane 1–3, all damage variables except ω_2 are zero, and the coefficient matrixes in Eqs. (12), (15) and (16) can be simplified as

$$(C)_{pq} = \begin{bmatrix} 2C_2\omega_2 & C_{14}\omega_2 & C_{20}\omega_2 & 0 \\ & 2C_5\omega_2 & C_{17}\omega_2 & 0 \\ & & 2C_8\omega_2 & 0 \\ & & & 2C_{11}\omega_2 \end{bmatrix}, \tag{18}$$

$$(e)_{pm} = \begin{bmatrix} e_2\omega_2 & e_{11}\omega_2 & e_{20}\omega_2 \\ e_5\omega_2 & e_{14}\omega_2 & e_{23}\omega_2 \\ e_8\omega_2 & e_{17}\omega_2 & e_{26}\omega_2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{19}$$

$$(\kappa)_{mn} = \begin{bmatrix} 2\kappa_2\omega_2 & \kappa_{11}\omega_2 & \kappa_{17}\omega_2 \\ & 2\kappa_5\omega_2 & \kappa_{14}\omega_2 \\ & & 2\kappa_8\omega_2 \end{bmatrix}. \tag{20}$$

Due to the fact that cracks are parallel to the coordinate plane 1-3, the effect of the damage on the plate stiffness in this coordinate plane 1–3 can be neglected. Then matrix (18) can be further simplified as

$$(C)_{pq} = \begin{bmatrix} 0 & C_{14}\omega_2 & C_{20}\omega_2 & 0 \\ & 2C_5\omega_2 & C_{17}\omega_2 & 0 \\ & & 0 & 0 \\ & & & 2C_{11}\omega_2 \end{bmatrix}. \tag{21}$$

Letting $\sigma_3 = 0$, the constitutive relations of the piezoelectric plate with damage for the plane stress problem are obtained as follows

$$\begin{aligned} \sigma_p &= [(C^0)_{pq} + (c)_{pq}]\varepsilon_q - [(e^0)_{pm} + (e)_{pm}]E_m, \\ D_m &= [(e^0)_{qm} + (e)_{qm}]^{\mathrm{T}}\varepsilon_q + [(\kappa^0)_{mn} + (\kappa)_{mn}]^{\mathrm{T}}E_m, \\ & p, q = 1, 2, 6, \quad m, n = 1, 2, 3, \end{aligned} \tag{22}$$

where

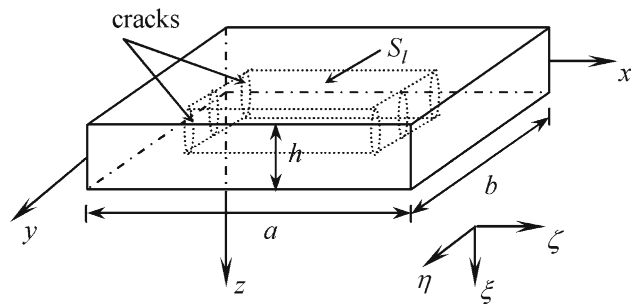

Fig. 2 Geometric configuration of a piezoelectric plate with transverse cracks

3 Free vibration of piezoelectric plates with damage

3.1 Basic equations of piezoelectric plates

Consider a piezoelectric plate with transverse cracks of thickness h, length a in the x-direction, width b in the y-direction

$$(C^0)_{pq} = \begin{bmatrix} 2C_1^0 - \dfrac{(C_6^0)^2}{2C_5^0} & C_2^0 - \dfrac{C_6^0C_7^0}{2C_5^0} & 0 \\ & 2C_3^0 - \dfrac{(C_7^0)^2}{2C_5^0} & 0 \\ & & 2C_4^0 \end{bmatrix} \underline{\underline{\mathrm{def}}} \begin{bmatrix} C_{11}^* & C_{12}^* & 0 \\ & C_{22}^* & 0 \\ & & C_{66}^* \end{bmatrix}, \tag{23}$$

$$(C)_{pq} = \begin{bmatrix} \dfrac{C_6^0C_{20}\omega_2}{C_5^0} & \left(C_{14} - \dfrac{C_7^0C_{20} + C_6^0C_{17}}{2C_5^0}\right)\omega_2 & 0 \\ & \left(2C_5 - \dfrac{C_7^0C_{17}}{C_5^0}\right)\omega_2 & 0 \\ & & 2C_{11}\omega_2 \end{bmatrix} \underline{\underline{\mathrm{def}}} \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ & \alpha_{22} & 0 \\ & & \alpha_{66} \end{bmatrix}\omega_2, \tag{24}$$

$$(e^0)_{pm} = \begin{bmatrix} e_1^0 - \dfrac{C_6^0e_3^3}{2C_5^0} & e_4^0 - \dfrac{C_6^0e_6^3}{2C_5^0} & e_7^0 - \dfrac{C_6^0e_9^3}{2C_5^0} \\ e_2^0 - \dfrac{C_7^0e_3^3}{2C_5^0} & e_5^0 - \dfrac{C_7^0e_6^3}{2C_5^0} & e_8^0 - \dfrac{C_7^0e_9^3}{2C_5^0} \\ 0 & 0 & 0 \end{bmatrix} \underline{\underline{\mathrm{def}}} \begin{bmatrix} e_{11}^* & e_{12}^* & e_{13}^* \\ e_{21}^* & e_{22}^* & e_{23}^* \\ 0 & 0 & 0 \end{bmatrix}, \tag{25}$$

$$(e)_{pm} = \begin{bmatrix} \left(e_2 - \dfrac{C_6^0e_8 + C_{20}e_3^0}{2C_5^0}\right)\omega_2 & \left(e_{11} - \dfrac{C_6^0e_{17} + C_{20}e_6^0}{2C_5^0}\right)\omega_2 & \left(e_{20} - \dfrac{C_6^0e_{26} + C_{20}e_9^0}{2C_5^0}\right)\omega_2 \\ \left(e_5 - \dfrac{C_7^0e_8 + C_{17}e_3^0}{2C_5^0}\right)\omega_2 & \left(e_{14} - \dfrac{C_6^0e_{17} + C_{17}e_6^0}{2C_5^0}\right)\omega_2 & \left(e_{23} - \dfrac{C_7^0e_{26} + C_{17}e_9^0}{2C_5^0}\right)\omega_2 \\ 0 & 0 & 0 \end{bmatrix} \underline{\underline{\mathrm{def}}} \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ 0 & 0 & 0 \end{bmatrix}\omega_2, \tag{26}$$

$$(\kappa^0)_{mn} = \begin{bmatrix} 2\kappa_1^0 + \dfrac{(e_3^0)^2}{2C_5^0} & \kappa_4^0 + \dfrac{e_3^0e_6^0}{2C_5^0} & \kappa_6^0 + \dfrac{e_3^0e_9^0}{2C_5^0} \\ & 2\kappa_2^0 + \dfrac{(e_6^0)^2}{2C_5^0} & \kappa_5^0 + \dfrac{e_6^0e_9^0}{2C_5^0} \\ & & 2\kappa_3^0 + \dfrac{(e_9^0)^2}{2C_5^0} \end{bmatrix} \underline{\underline{\mathrm{def}}} \begin{bmatrix} \kappa_{11}^* & \kappa_{12}^* & \kappa_{13}^* \\ & \kappa_{22}^* & \kappa_{23}^* \\ & & \kappa_{33}^* \end{bmatrix}, \tag{27}$$

$$(\kappa)_{mn} = \begin{bmatrix} \left(2\kappa_2 + \dfrac{e_3^0e_8}{C_5^0}\right)\omega_2 & \left(\kappa_{11} + \dfrac{e_3^0e_{17} + e_6^0e_8}{2C_5^0}\right)\omega_2 & \left(\kappa_{17} + \dfrac{e_3^0e_{26} + e_9^0e_8}{2C_5^0}\right)\omega_2 \\ & \left(2\kappa_5 + \dfrac{e_6^0e_{17}}{C_5^0}\right)\omega_2 & \left(\kappa_{14} + \dfrac{e_6^0e_{26} + e_9^0e_{17}}{2C_5^0}\right)\omega_2 \\ & & \left(2\kappa_{18} + \dfrac{e_9^0e_{26}}{C_5^0}\right)\omega_2 \end{bmatrix} \underline{\underline{\mathrm{def}}} \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ & \gamma_{22} & \gamma_{23} \\ & & \gamma_{33} \end{bmatrix}\omega_2. \tag{28}$$

as shown in Fig. 2. The reference surface defined by $z = 0$ is the middle surface of the underformed plate. The principal directions of the material are assumed to coincide with the coordinates x, y and z.

According to the Kirchhoff hypothesis of plate, the displacement components $u_i (i = 1, 2, 3)$ in the x, y, z directions can be expressed as

$$\begin{aligned} u_1(x, y, z, t) &= -zw_{,x}, \\ u_2(x, y, z, t) &= -zw_{,y}, \\ u_3(x, y, z, t) &= w(x, y, t), \end{aligned} \tag{29}$$

where a comma denotes partial differentiation with respect to the corresponding coordinate.

The strain–displacement relations of the piezoelectric plate can be written as

$$\begin{aligned} \varepsilon_x &= -zw_{,xx}, \\ \varepsilon_y &= -zw_{,yy}, \\ \gamma_{xy} &= -2zw_{,xy}. \end{aligned} \tag{30}$$

The relations between the electric vectors $E_i (i = 1, 2, 3)$ and the electric potential ϕ in the Cartesian coordinate system are defined by

$$E_i = -\phi_{,i}, \quad i = 1, 2, 3. \tag{31}$$

Suppose that the damage variables remain unchanged through the thickness of plate and denote M_x, M_y, M_{xy} as the distributed moments in the piezoelectric plate, according to the classical plate theory, the following equations can be obtained

$$[M_x, M_y, M_{xy}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma_x, \sigma_y, \sigma_{xy}] z \mathrm{d}z. \tag{32}$$

The equation of motion of the plate can be written as

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + q(x, y, t) = \rho h w_{,tt}, \tag{33}$$

where $q(x, y, t)$ is the intensity of the transverse force and ρ is mass density.

The electric variables must also satisfy the Maxwell equations [19]. This condition can be satisfied approximately by letting the integration of the divergence of the electric flux across the thickness of the piezoelectric plate to vanish for any x and y, that is

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z} \right) \mathrm{d}z = 0. \tag{34}$$

Consider the free vibration of the piezoelectric plate and suppose that the distribution of electric potential along the thickness direction in the piezoelectric plate is simulated by a sinusoidal function, so the displacement field and the electric potential field can be written as [15]

$$\begin{aligned} w(x, y, t) &= W(x, y) \sin(\Omega t), \\ \phi(x, y, z, t) &= \varphi(x, y) \sin \pi (z + h/2) \sin(\Omega t), \end{aligned} \tag{35}$$

where Ω is the natural frequency of the plate.

The dimensionless parameters are defined as

$$\begin{aligned} &\zeta = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad \xi = \frac{z}{h}, \\ &\lambda_1 = \frac{h}{a}, \quad \lambda_2 = \frac{h}{b}, \quad \overline{W} = \frac{W}{h}, \\ &\overline{C}_{11} = \frac{C_{11}^*}{C_{\mathrm{ref}}}, \quad \overline{C}_{12} = \frac{C_{12}^*}{C_{\mathrm{ref}}}, \quad \overline{C}_{22} = \frac{C_{22}^*}{C_{\mathrm{ref}}}, \\ &\overline{C}_{66} = \frac{C_{66}^*}{C_{\mathrm{ref}}}, \quad \bar{e}_{13} = \frac{e_{13}^*}{e_{\mathrm{ref}}}, \quad \bar{e}_{23} = \frac{e_{23}^*}{e_{\mathrm{ref}}}, \\ &\overline{\alpha}_{11} = \frac{\alpha_{11}}{C_{\mathrm{ref}}}, \quad \overline{\alpha}_{12} = \frac{\alpha_{12}}{C_{\mathrm{ref}}}, \quad \overline{\alpha}_{22} = \frac{\alpha_{22}}{C_{\mathrm{ref}}}, \\ &\overline{\alpha}_{66} = \frac{\alpha_{66}}{C_{\mathrm{ref}}}, \quad \overline{\beta}_{13} = \frac{\beta_{13}}{e_{\mathrm{ref}}}, \quad \overline{\beta}_{23} = \frac{\beta_{23}}{e_{\mathrm{ref}}}, \\ &\overline{\kappa}_{11} = \frac{\kappa_{11}^* C_{\mathrm{ref}}}{(e_{\mathrm{ref}})^2}, \quad \overline{\kappa}_{12} = \frac{\kappa_{12}^* C_{\mathrm{ref}}}{(e_{\mathrm{ref}})^2}, \quad \overline{\kappa}_{22} = \frac{\kappa_{22}^* C_{\mathrm{ref}}}{(e_{\mathrm{ref}})^2}, \\ &\overline{\kappa}_{33} = \frac{\kappa_{33}^* C_{\mathrm{ref}}}{(e_{\mathrm{ref}})^2}, \quad \overline{\gamma}_{11} = \frac{\gamma_{11} C_{\mathrm{ref}}}{(e_{\mathrm{ref}})^2}, \quad \overline{\gamma}_{12} = \frac{\gamma_{12} C_{\mathrm{ref}}}{(e_{\mathrm{ref}})^2}, \\ &\overline{\gamma}_{22} = \frac{\gamma_{22} C_{\mathrm{ref}}}{(e_{\mathrm{ref}})^2}, \quad \overline{\gamma}_{33} = \frac{\gamma_{33} C_{\mathrm{ref}}}{(e_{\mathrm{ref}})^2}, \quad \bar{\varphi} = \frac{\varphi e_{\mathrm{ref}}}{h C_{\mathrm{ref}}}, \\ &\tau = \frac{t}{a}\sqrt{\frac{C_{\mathrm{ref}}}{\rho}}, \quad \varpi = \Omega a \sqrt{\frac{\rho}{C_{\mathrm{ref}}}}, \end{aligned} \tag{36}$$

where $C_{\mathrm{ref}} = C_{11}^*$, $e_{\mathrm{ref}} = e_{13}^*$.

Using Eqs. (22) and (30)–(33), the dimensionless equations governing the motion of the piezoelectric plate can be written as

$$\begin{aligned} &-\frac{2}{\pi}[\overline{\gamma}_{11}\lambda_1^2\omega_{2,\zeta}\overline{\varphi}_{,\zeta} + \lambda_1^2(\overline{\kappa}_{11} + \overline{\gamma}_{11}\omega_2)\overline{\varphi}_{,\zeta\zeta} \\ &\quad + \lambda_1\lambda_2\overline{\gamma}_{12}\omega_{2,\zeta}\overline{\varphi}_{,\eta} + \lambda_1\lambda_2(\overline{\kappa}_{12} + \overline{\gamma}_{12}\omega_2)\overline{\varphi}_{,\zeta\eta} \\ &\quad + \lambda_1\lambda_2\overline{\gamma}_{12}\omega_{2,\eta}\overline{\varphi}_{,\zeta} + \lambda_1\lambda_2(\overline{\kappa}_{21} + \overline{\gamma}_{12}\omega_2)\overline{\varphi}_{,\zeta\eta} \\ &\quad + \lambda_2^2\overline{\gamma}_{22}\omega_{2,\eta}\overline{\varphi}_{,\eta} + \lambda_2^2(\overline{\kappa}_{22} + \overline{\gamma}_{22}\omega_2)\overline{\varphi}_{,\eta\eta}] \\ &\quad - (\overline{e}_{13} + \overline{\beta}_{13}\omega_2)\lambda_1^2\overline{W}_{,\zeta\zeta} - (\overline{e}_{23} + \overline{\beta}_{23}\omega_2)\lambda_2^2\overline{W}_{,\eta\eta} \\ &\quad + 2\pi(\overline{\kappa}_{33} + \overline{\gamma}_{33}\omega_2)\overline{\varphi} = 0, \\ &-\overline{\alpha}_{11}\pi\lambda_1^4\omega_{2,\zeta\zeta}\overline{W}_{,\zeta\zeta} - 2\overline{\alpha}_{11}\pi\lambda_1^4\omega_{2,\zeta}\overline{W}_{,\zeta\zeta\zeta} \\ &\quad - (\overline{C}_{11} + \overline{\alpha}_{11})\pi\lambda_1^4\omega_2\overline{W}_{,\zeta\zeta\zeta\zeta} - \pi\lambda_1^2\lambda_2^2\overline{\alpha}_{12}\omega_{2,\zeta\zeta}\overline{W}_{,\eta\eta} \\ &\quad - 2\overline{\alpha}_{12}\pi\lambda_1^2\lambda_2^2\omega_{2,\zeta}\overline{W}_{,\zeta\eta\eta} - (\overline{C}_{12} + \overline{\alpha}_{12}\omega_2)\pi\lambda_1^2\lambda_2^2\overline{W}_{,\zeta\zeta\eta\eta} \\ &\quad - 4[\pi\lambda_1^2\lambda_2^2\overline{\alpha}_{66}\omega_{2,\zeta\eta}\overline{W}_{,\zeta\eta} + \pi\lambda_1^2\lambda_2^2\overline{\alpha}_{66}\omega_{2,\zeta\eta}\overline{W}_{,\zeta\eta\eta} \\ &\quad + \pi\lambda_1^2\lambda_2^2\overline{\alpha}_{66}\omega_{2,\zeta\eta}\overline{W}_{,\zeta\zeta\eta} + (\overline{C}_{66} + \overline{\alpha}_{66}\omega_2)\pi\lambda_1^2\lambda_2^2\overline{W}_{,\zeta\zeta\eta\eta}] \\ &\quad - \overline{\alpha}_{12}\pi\lambda_1^2\lambda_2^2\omega_{2,\eta\eta}\overline{W}_{,\zeta\zeta} - 2\overline{\alpha}_{12}\pi\lambda_1^2\lambda_2^2\omega_{2,\eta}\overline{W}_{,\zeta\zeta\eta} \\ &\quad - (\overline{C}_{12} + \overline{\alpha}_{12}\omega_2)\pi\lambda_1^2\lambda_2^2\overline{W}_{,\zeta\zeta\eta\eta} - \pi\lambda_2^4\overline{\alpha}_{12}\omega_{2,\eta\eta}\overline{W}_{,\eta\eta} \end{aligned} \tag{37}$$

$$-2\pi\lambda_2^4\overline{\alpha}_{22}\omega_{2,\eta}\overline{W}_{,\eta\eta\eta} - (\overline{C}_{22} + \overline{\alpha}_{22}\omega_2)\pi\lambda_2^4\overline{W}_{,\eta\eta\eta\eta}$$
$$-24[\lambda_1^2\overline{\beta}_{13}\omega_{2,\zeta\zeta}\bar{\varphi} + 2\lambda_1^2\overline{\beta}_{13}\omega_{2,\eta}\bar{\varphi}_{,\zeta}$$
$$+(\bar{e}_{13} + \overline{\beta}_{13}\omega_2)\bar{\varphi}_{,\zeta\zeta}\lambda_1^2$$
$$+\frac{\lambda_2^4}{\lambda_1}\overline{\beta}_{23}\omega_{2,\eta\eta}\bar{\varphi} + 2\lambda_2^4\overline{\beta}_{23}\omega_{2,\eta}\overline{\varphi}_{,\eta}$$
$$+(\bar{e}_{23} + \overline{\beta}_{23}\omega_2)\lambda_2^2\overline{\varphi}_{,\eta\eta}] + 12\varpi^2\pi\lambda_1\overline{W} = 0. \tag{38}$$

3.2 Solution methodology

For the piezoelectric plate in the simply supported condition and with zero electrical potential along the plate edges, the solution of Eqs. (34) and (35) is sought in the following separable form

$$\overline{W}(\zeta,\eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\sin(m\pi\zeta)\sin(n\pi\eta),$$
$$\overline{\varphi}(\zeta,\eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varphi_{mn}\sin(m\pi\zeta)\sin(n\pi\eta). \tag{39}$$

We assume that the damage variable ω_2 takes a fixed value, that is, damage evolution is not considered and damage exists only in a certain local rectangle region $S_l(a_1 \le \zeta \le a_2, b_1 \le \eta \le b_2)$. Substituting Eq. (39) into Eq. (38) and multiplying them by $\sin(m\pi\zeta)\sin(n\pi\eta)$, respectively, then integrating them from 0 to 1 with respect to ζ and η, a set of homogeneous linear equations in term of W_{mn} and φ_{mn} is obtained as follows

$$\begin{aligned}
&[2\pi^2m^2\lambda_1^2\overline{\kappa}_{11} + 2\pi^2n^2\lambda_2^2\overline{\kappa}_{22} + 2\pi^2\overline{\kappa}_{33}\\
&\quad + (8\pi^2m^2\lambda_1^2\overline{\gamma}_{11}\omega_2 + 8\pi^2n^2\lambda_2^2\overline{\gamma}_{22}\omega_2\\
&\quad + 8\pi^2\overline{\gamma}_{33}\omega_2)\Theta]\varphi_{mn} + [\pi^3m^2\lambda_1^2\bar{e}_{13}\\
&\quad + \pi^3n^2\lambda_2^2\bar{e}_{23} + (4\pi\lambda_1^2\overline{\beta}_{13}\omega_2\\
&\quad + 4\pi\lambda_2^2\overline{\beta}_{23}\omega_2)\Theta]W_{mn} = 0,\\
&-[\pi^5m^4\lambda_1^4\overline{C}_{11} + 2\pi^5m^2n^2\lambda_1^2\lambda_2^2\overline{C}_{12}\\
&\quad + \pi^5n^4\lambda_2^4\overline{C}_{22} + 4\pi^5m^2n^2\lambda_1^2\lambda_2^2\overline{C}_{66}\\
&\quad + 12\pi\lambda_1^2\varpi^2 + (4\pi^5m^4\lambda_1^4\overline{\alpha}_{11}\omega_2\\
&\quad + 8\pi^5m^2n^2\lambda_1^2\lambda_2^2\overline{\alpha}_{12}\omega_2\\
&\quad + 16\pi^5m^2n^4\lambda_1^2\lambda_2^2\overline{\alpha}_{66}\omega_2\\
&\quad + 4\pi^5n^4\lambda_2^4\overline{\alpha}_{22}\omega_2)\Theta]W_{mn}\\
&\quad + [24\pi^2m^2\lambda_1^2\bar{e}_{13} + 24\pi^2n^2\lambda_2^2\bar{e}_{23}\\
&\quad + (96\pi^2m^2\lambda_1^2\overline{\beta}_{13}\omega_2\\
&\quad + 96\pi^2n^2\lambda_n^2\overline{\beta}_{23}\omega_2)\Theta]\varphi_{mn} = 0,
\end{aligned} \tag{40}$$

where $\Theta = \int_{a_1}^{a_2}\int_{b_1}^{b_2}[\sin(m\pi\zeta)\sin(n\pi\xi)]^2\mathrm{d}\xi\mathrm{d}\zeta$.

For a non-trivial solution of Eqs. (40), the determinant of the coefficient matrix of the equations must be equal to zero, i.e.,

$$|R| = 0, \tag{41}$$

where $R_{ij}(i, j = 1, 2)$ can be easily determined from Eqs. (40). From Eq. (41), the vibrating frequency of the simply supported piezoelectric rectangular plate with damage can be obtained.

3.3 Numerical results

In the following numerical calculations, the piezoelectric material is assumed to be the ceramic PZT4 and the material constants are

$$\begin{aligned}
&C_{11}^* = 13.2\times10^{10}\mathrm{N/m^2}, \quad C_{12}^* = 7.1\times10^{10}\mathrm{N/m^2},\\
&C_{13}^* = 7.3\times10^{10}\mathrm{N/m^2}, \quad C_{33}^* = 11.5\times10^{10}\mathrm{N/m^2},\\
&C_{44}^* = 2.6\times10^{10}\mathrm{N/m^2}, \quad C_{66}^* = 3.0\times10^{10}\mathrm{N/m^2},\\
&e_{31}^* = -4.1\mathrm{C/m^2}, \quad e_{33}^* = 14.1\mathrm{C/m^2},\\
&\kappa_{11}^* = \kappa_{22}^* = 7.142\times10^{-9}\mathrm{F/m}, \quad \kappa_{33}^* = 5.841\times10^{-9}\mathrm{F/m},\\
&\rho = 7.5\times10^3\mathrm{kg/m^3}.
\end{aligned} \tag{42}$$

The dimensionless vibrating frequencies $\varpi_{mn}(m, n = 1, 2, 3)$ without damage for a thin piezoelectric rectangular plate with $\lambda_1 = 0.01, \lambda_2 = 0.02$ under the close-circuited condition are given in Table 1. The results in Table 1 agree with those of the 3D analysis in Ref. [20]. From Table 1, it can be seen that the results under the Kirchhoff hypothesis for plate are larger than the corresponding ones from the 3D analysis, a point which must be considered in engineering design.

The effect of the side-to-thickness ratio λ_2 on the dimensionless vibrating frequencies of the square piezoelectric plate with damage or without damage is shown in Fig. 3. The material parameters related to damage in Fig. 3 are $\overline{\alpha}_{11} = \overline{\alpha}_{22} = \overline{\alpha}_{12} = \overline{\alpha}_{66} = -0.2$, $\overline{\beta}_{13} = \overline{\beta}_{23} = -0.2$, $\overline{\gamma}_{11} = \overline{\gamma}_{22} = \overline{\gamma}_{33} = -0.2$, $\omega_2 = 0.5$. The centroid of the damaged region center in the piezoelectric plate is ($\zeta = 0.5, \xi = 0.5$) and the ratio of damaged region S_l to the total area of the piezoelectric plate S is $\mu = 25\%$. From Fig. 3, it can be seen that the dimensionless vibrating frequencies increase with the value of λ_2, as in agreement with the results of the elastic plate without considering piezoelectric effect. Moreover, it is demonstrated that the vibrating frequencies of the piezoelectric plate with damage are smaller than the corresponding ones without damage. Meanwhile, when the side-to-thickness ratio λ_2 increases, the difference of the dimensionless vibrating frequencies with damage and without damage becomes more and more significant.

Shown in Fig. 4 is the effect of the size of damaged region $S_l(\mu = S_l/S)$ on the dimensionless vibrating frequency ϖ_{11}

Table 1 The dimensionless vibrating frequencies of the piezoelectric rectangular plate without damage

	ϖ_{11}	ϖ_{12}	ϖ_{13}	ϖ_{21}	ϖ_{22}	ϖ_{23}	ϖ_{31}	ϖ_{32}	ϖ_{33}
Present work	0.1426	0.4845	1.0543	0.2290	0.5706	1.1403	0.3731	0.7144	1.2839
Ref. [18]	0.1287	0.4368	0.9476	0.2056	0.5132	1.0237	0.3338	0.6405	1.1492

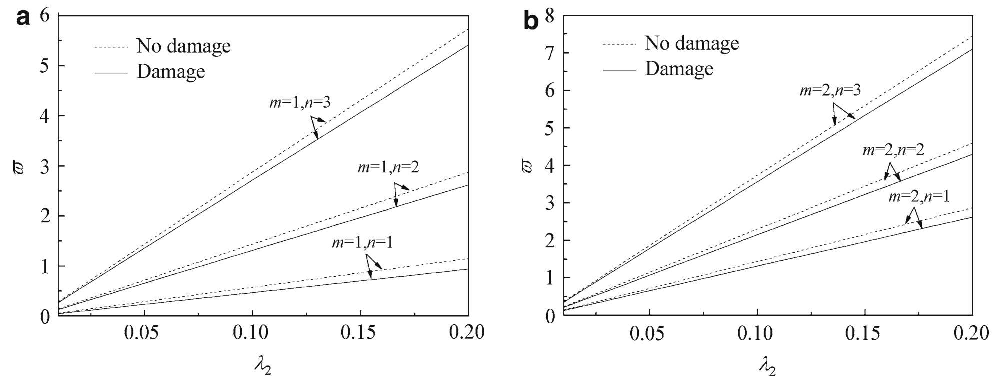

Fig. 3 Effect of the side-to-thickness ratio on the dimensionless vibrating frequencies of the piezoelectric square plate with damage or without damage

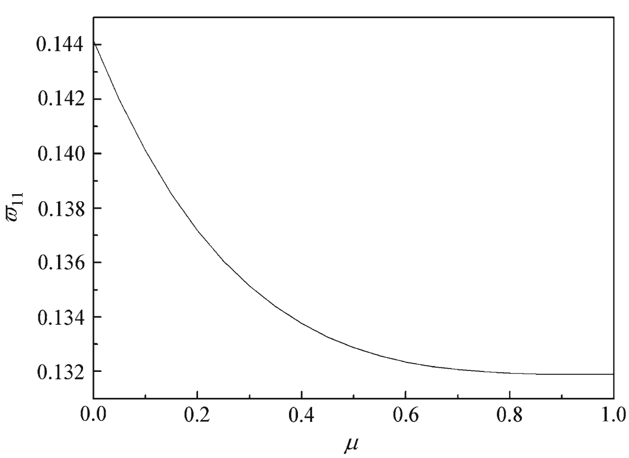

Fig. 4 Effect of size of the damaged region on the frequency ϖ_{11} of the piezoelectric plate

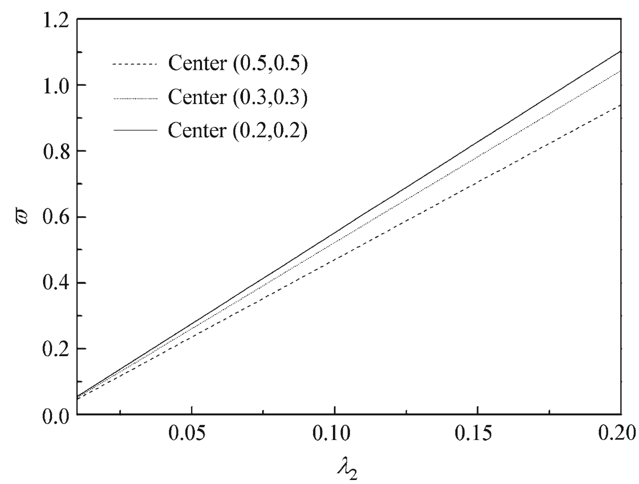

Fig. 5 Effect of the location of damaged region on the frequency ϖ_{11} of the piezoelectric plate

of the piezoelectric rectangular plate with $\lambda_1 = 0.01, \lambda_2 = 0.02$. The material parameters related to damage in Fig. 5 are the same as in Fig. 4. It can be concluded that the dimensionless vibrating frequencies of the plate with damage decrease with the increase of the size of damaged region. And when the damaged region increases moderately, the decrease rate of the frequency ϖ_{11} is small.

Shown in Fig. 5 is the effect of the location of the damaged region on the frequency ϖ_{11} of the piezoelectric plate with $\lambda_1 = 0.01, \lambda_2 = 0.02$. The material parameters related to damage in Fig. 5 are the same as in Fig. 4. It can be seen that when the centroid of the damaged center moves away from the mid-point of the plate, the value of the frequency ϖ_{11} increases. It can be conclude that the stiffness of plate reduces with the decrease of the distance of the centroid of the damaged region from the mid-point of the plate.

4 Conclusions

By using the Telreja's tensor valued internal state damage variables as well as the Helmhotlz free energy of piezoelectric material, a new constitutive model for piezoelectric material solids containing distributed cracks is established. This model is successfully used to analyze the static-dynamic behaviors of the piezoelectric structures. A special case of the free vibration of piezoelectric plate with damage is investigated. Numerical results show that the damage has a significant effect on the vibrating frequencies of the piezoelectric plate. The vibrating frequencies increase with the side-to-thickness ratio, as in agreement with the results of the elastic plate without considering piezoelectric effect. And, the increase of the size of the damaged region will decrease the stiffness of the structure and the vibrating frequencies. When the distance of the centroid of the damaged region to the mid-point of the plate increases, the vibrating frequencies will also increase.

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